Math 221: LINEAR ALGEBRA

Chapter 2. Matrix Algebra
§2-7. LU Factorization

Le Chen[1]

Emory University, 2020 Fall

(last updated on 10/26/2020)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

LU Factorization

Why do we need LU Factorization?

Finding the LU

Multiplier Method

LU Factorization

LU Factorization

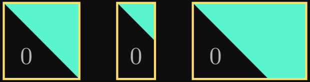
Definition

A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.

LU Factorization

Definition

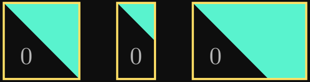
A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.



LU Factorization

Definition

A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.

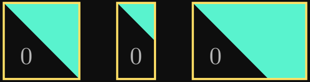


A lower triangular matrix is defined similarly, as a matrix for which all entries above the main diagonal are equal to zero.

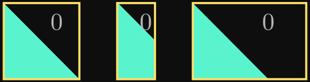
LU Factorization

Definition

A matrix $A = [a_{ij}]$ is called upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries below the main diagonal equal 0.



A lower triangular matrix is defined similarly, as a matrix for which all entries above the main diagonal are equal to zero.



An LU factorization of a matrix A is written

$$A = LU$$

where L is lower triangular matrix and U is upper triangular.

An LU factorization of a matrix A is written

$$A = LU$$

where L is lower triangular matrix and U is upper triangular.

We often require either L or U to have only 1's on the main diagonal.

An LU factorization of a matrix A is written

$$A = LU$$

where L is lower triangular matrix and U is upper triangular.

We often require either L or U to have only 1's on the main diagonal.

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

Why do we need LU Factorization?

Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Suppose we wish to find all solutions $\vec{x}$ to the system $A\vec{x} = B$. The LU factorization of A can assist in this process.

Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Suppose we wish to find all solutions $\vec{x}$ to the system $A\vec{x} = B$. The LU factorization of A can assist in this process.

Consider the following reduction:

$$\begin{aligned} A\vec{x} &= B \\ (LU)\vec{x} &= B \\ L(U\vec{x}) &= B \\ L\vec{y} &= B \end{aligned}$$

Why do we need LU Factorization?

The LU factorization often helps to quickly solve equations of the form $A\vec{x} = \vec{b}$.

Suppose we wish to find all solutions $\vec{x}$ to the system $A\vec{x} = B$. The LU factorization of A can assist in this process.

Consider the following reduction:

$$\begin{aligned} A\vec{x} &= B \\ (LU)\vec{x} &= B \\ L(U\vec{x}) &= B \\ L\vec{y} &= B \end{aligned}$$

Therefore, if we can solve $L\vec{y} = B$ for $\vec{y}$, then all that remains is to solve $U\vec{x} = \vec{y}$ for $\vec{x}$.

Example

Find all solutions to

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 10 & 5 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Example

Find all solutions to

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 10 & 5 & 1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Solution

Using a method of your choice, verify that the LU factorization of A gives

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

The solution is $\vec{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Solution (continued)

Let $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and solve $L\vec{y} = \vec{b}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

The solution is $\vec{y} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

Now we solve $U\vec{x} = \vec{y}$.

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

Solution (continued)

Multiplying and solving (or finding the reduced row-echelon form), the general solution is given by

$$\vec{x} = \begin{bmatrix} -12 \\ 2 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 13 \\ -3 \\ -2 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

■

Finding the LU Factorization

Finding the LU Factorization

An LU factorization can be found for a matrix A provided that the row-echelon form of A can be calculated without interchanging rows. In this case, we call that A can be lower reduced.

Finding the LU Factorization

An LU factorization can be found for a matrix A provided that the row-echelon form of A can be calculated without interchanging rows. In this case, we call that A can be lower reduced.

Example

Determine if the LU factorization of A exists, and if so, find it.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Finding the LU Factorization

An LU factorization can be found for a matrix A provided that the row-echelon form of A can be calculated without interchanging rows. In this case, we call that A can be lower reduced.

Example

Determine if the LU factorization of A exists, and if so, find it.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution

Because the row-echelon form can be obtained without interchanging rows:

Finding the LU Factorization

An LU factorization can be found for a matrix A provided that the row-echelon form of A can be calculated without interchanging rows. In this case, we call that A can be lower reduced.

Example

Determine if the LU factorization of A exists, and if so, find it.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution

Because the row-echelon form can be obtained without interchanging rows:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} \xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow{r_3 + r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

the LU factorization exists, or A can be lower reduced.

Solution (continued)

We proceed to finding L and U. Assign variables to the unknown entries and multiply.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

Solution (continued)

We proceed to finding L and U. Assign variables to the unknown entries and multiply.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

Solving each entry will give us values for the unknown entries.

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

We see easily that $a = 1, d = 1$, and $e = 2$. Continuing to solve the first column gives $x = 2, y = 1$. The other values are calculated as follows.

Solution (continued)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a & d & e \\ ax & dx + b & ex + f \\ ay & dy + bz & ey + fz + c \end{bmatrix}$$

We see easily that $a = 1, d = 1$, and $e = 2$. Continuing to solve the first column gives $x = 2, y = 1$. The other values are calculated as follows.

$$\begin{aligned} dx + b &= 3 \\ (1)(2) + b &= 3 \\ b &= 1 \end{aligned}$$

$$\begin{aligned} ex + f &= 0 \\ (2)(2) + f &= 0 \\ f &= -4 \end{aligned}$$

$$\begin{aligned} dy + bz &= 0 \\ (1)(1) + (1)z &= 0 \\ z &= -1 \end{aligned}$$

$$\begin{aligned} ey + fz + c &= 5 \\ (2)(1) + (-4)(-1) + c &= 5 \\ c &= -1 \end{aligned}$$

Solution (continued)

Therefore,

$$
\begin{array}{c}
L \\
\| \\
\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \\
\| \\
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
\end{array}
$$

Solution (continued)

Therefore,

$$\begin{array}{ccc} L & & U \\ \| & & \| \\ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} & & \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \\ \| & & \| \\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} \end{array}$$

Solution (continued)

Therefore,

$$
\begin{array}{ccc}
L & & U \\
\| & & \| \\
\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} & & \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \\
\| & & \| \\
\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} & & \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}
\end{array}
$$

You should multiply these and check that they equal A!

Solution (continued)

If you want the diagonal terms of U to be all 1's:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} 1 & 0 & -0 \\ 2 & 1 & -0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ -0 & -0 & 1 \end{bmatrix}$$

■

Multiplier Method

Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution

First, write A as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$||$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiplier Method

The following process for finding L and U, called the multiplier method, can be more efficient.

Example

Find the LU factorization of $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution

First, write A as

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$||$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Solution (continued)

To do so, we use row operations to remove the entries of A below the main diagonal. For every operation we apply to A (the matrix on the right), we apply the inverse operation to the identity matrix (on the left). This ensures the product remains the same.

Solution (continued)

To do so, we use row operations to remove the entries of A below the main diagonal. For every operation we apply to A (the matrix on the right), we apply the inverse operation to the identity matrix (on the left). This ensures the product remains the same.

The first step is to add (-2) times the first row of A to the second row. To preserve the product, add (2) times the second column to the first column, for the matrix on the left.

$$\|$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

$$\|$$

$$c_1 + 2c_2 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 1 & 0 & 5 \end{bmatrix} r_2 - 2r_1 \to r_2$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} r_3 - r_1 \to r_3$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} r_3 - r_1 \to r_3$$

$$c_2 - c_3 \to c_2 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} r_3 + r_2 \to r_3$$

Solution (continued)

We proceed in the same way.

$$c_1 + c_3 \to c_1 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & -1 & 3 \end{bmatrix} r_3 - r_1 \to r_3$$

$$c_2 - c_3 \to c_2 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix} r_3 + r_2 \to r_3$$

At this point we have a lower triangular matrix L on the left, and an upper triangular matrix U on the right so we are done. You can (and should!) check that this product equals A.

If you want the diagonal terms of U to be all 1's:

$$-1 \times c_3 \to c_3 \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} - 1 \times r_3 \to r_3$$

■

Problem

Use the multiplier method to verify the LU factorization for

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix}$$

Problem

Use the multiplier method to verify the LU factorization for

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 13 & 5 \\ -2 & -7 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

■

The multiplier method can be simplified using so-called LU-Algorithm.
See Examples 2.7.2 – 2.7.4 on the book.