Math 221: LINEAR ALGEBRA

Chapter 3. Determinants and Diagonalization

§3-2. Determinants and Matrix Inverses

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Determinants and Matrix Inverses

Adjugates

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Determinants and Matrix Inverses

Theorem (Product Theorem)

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$

Theorem (Determinant of Matrix Inverse)

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}.$$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

$$= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c).$$

Therefore, A is invertible for all $c \neq 0, 3, -3$.

Theorem (Determinant of Matrix Transpose)

If A is an $n \times n$ matrix, then $\det(A^T) = \det A$.

Proof.

1. This is trivially true for all elementary matrices.
2. If A is not invertible, then neither is A^T (why?). Hence, $\det A = 0 = \det A^T$.
3. If A is invertible, then $A = E_k E_{k-1} \cdots E_2 E_1$. Hence, by case 1,

$$\det A = \cdots = \det A^T.$$

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Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

First,

$$\begin{aligned} \det(2A^{-1}) &= -4 \\ 2^3 \det(A^{-1}) &= -4 \\ \frac{1}{\det A} &= \frac{-4}{8} = -\frac{1}{2} \end{aligned}$$

Therefore, $\det A = -2$.

Example (continued)

Now,

$$\begin{aligned} \det(A^3(B^{-1})^T) &= -4 \\ (\det A)^3 \det(B^{-1}) &= -4 \\ (-2)^3 \det(B^{-1}) &= -4 \\ (-8)\det(B^{-1}) &= -4 \\ \frac{1}{\det B} &= \frac{-4}{-8} = \frac{1}{2} \end{aligned}$$

Therefore, $\det B = 2$.

Example

Suppose A, B and C are 4×4 matrices with

$$\det \mathrm{A} = -1, \det \mathrm{B} = 2, \quad \text{and} \quad \det \mathrm{C} = 1.$$

Find $\det(2\mathrm{A}^2(\mathrm{B}^{-1})(\mathrm{C}^{\mathrm{T}})^3\mathrm{B}(\mathrm{A}^{-1}))$.

$$\begin{aligned}
\det(2\mathrm{A}^2(\mathrm{B}^{-1})(\mathrm{C}^{\mathrm{T}})^3\mathrm{B}(\mathrm{A}^{-1})) &= 2^4(\det \mathrm{A})^2 \frac{1}{\det \mathrm{B}}(\det \mathrm{C})^3(\det \mathrm{B})\frac{1}{\det \mathrm{A}} \\
&= 16(\det \mathrm{A})(\det \mathrm{C})^3 \\
&= 16 \times (-1) \times 1^3 \\
&= -16.
\end{aligned}$$

Example

A square matrix A is orthogonal if and only if $A^T = A^{-1}$. What are the possible values of $\det A$ if A is orthogonal?

Since $A^T = A^{-1}$,

$$\begin{aligned} \det A^T &= \det(A^{-1}) \\ \det A &= \frac{1}{\det A} \\ (\det A)^2 &= 1 \end{aligned}$$

Assuming A is a real matrix, this implies that $\det A = \pm 1$, i.e., $\det A = 1$ or $\det A = -1$.

Adjugates

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of A defined as

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$\begin{aligned} A(\mathrm{adj}A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (\det A)I_2 \end{aligned}$$

Furthermore, if $\det A \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A.$$

Definition

If A is an $n \times n$ matrix, then

$$\mathrm{adj}A = \left[\ c_{ij}(A)\ \right]^T,$$

where $c_{ij}(A)$ is the (i,j)-cofactor of A, i.e., $\mathrm{adj}A$ is the transpose of the cofactor matrix (matrix of cofactors).

Reminder. $c_{ij}(A) = (-1)^{i+j}\det(A_{ij})$.

Example

Find adjA when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution.

$$\text{adj}A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$$

Notice that

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix} \\ &= \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix} \end{aligned}$$

Example (continued)

Also,

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 \\ 19 & 0 & 22 \\ 3 & 0 & -6 \end{vmatrix} \\ &= (-1)\begin{vmatrix} 19 & 22 \\ 3 & -6 \end{vmatrix} \\ &= 180, \end{aligned}$$

so in this example, we see that

$$A(\mathrm{adj}A) = (\det A)I$$

Theorem (The Adjugate Formula)

If A is an $n \times n$ matrix, then

$$A(\operatorname{adj}A) = (\det A)I = (\operatorname{adj}A)A.$$

Furthermore, if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}A.$$

Remark

Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Example

For an $n \times n$ matrix A, show that $\det(\text{adj}A) = (\det A)^{n-1}$.

Using the adjugate formula,

$$\begin{aligned} A(\text{adj}A) &= (\det A)I \\ \det(A(\text{adj}A)) &= \det((\det A)I) \\ (\det A) \times \det(\text{adj}A) &= (\det A)^n(\det I) \\ (\det A) \times \det(\text{adj}A) &= (\det A)^n \end{aligned}$$

If $\det A \neq 0$, then divide both sides of the last equation by $\det A$:

$$\det(\text{adj}A) = (\det A)^{n-1}.$$

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj}A) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj}A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of ($\star$))

We will prove ($\star$) by contradiction. Indeed, if $\det A = 0$, then

$$A(\mathrm{adj}A) = (\det A)I = (0)I = O,$$

i.e., $A(\mathrm{adj}A)$ is the zero matrix. If $\det(\mathrm{adj}A) \neq 0$, then $\mathrm{adj}A$ would be invertible, and $A(\mathrm{adj}A) = 0$ would imply $A = O$. However, if $A = O$, then $\mathrm{adj}A = 0$ and is not invertible, and thus has determinant equal to zero, i.e., $\det(\mathrm{adj}A) = 0$, (a contradiction!) Therefore, $\det(\mathrm{adj}A) = 0$, i.e., ($\star$) is true.

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Problem

Let A and B be $n \times n$ matrices. Show that $\det(A + B^T) = \det(A^T + B)$.

Solution

Notice that

$$(A + B^T)^T = A^T + (B^T)^T = A^T + B.$$

Since a matrix and it's transpose have the same determinant

$$\begin{aligned} \det(A + B^T) &= \det((A + B^T)^T) \\ &= \det(A^T + B). \end{aligned}$$

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

(a) If $\mathrm{adj}(A)$ exists, then A is invertible.

(c) If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.

Example

Prove or give a counterexample to the following statement:

$$\text{If } \det A = 1, \text{ then } \mathrm{adj} A = A.$$

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $A\vec{x} = \vec{b}$ can be given in terms of determinants of matrices.

Theorem (Cramer's Rule)

Let A be an $n \times n$ invertible matrix, the solution to the system $A\vec{x} = \vec{b}$ of n equations in teh variables $x_1, x_2 \cdots x_n$ is given by

$$x_1 = \frac{\det\left(A_1(\vec{b})\right)}{\det A}, \quad x_2 = \frac{\det\left(A_2(\vec{b})\right)}{\det A}, \quad \cdots, \quad x_n = \frac{\det\left(A_n(\vec{b})\right)}{\det A}$$

where, for each j, the matrix $A_j(\vec{b})$ is obtained from A by replacing column j with $\vec{b}$:

$$A_j(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix}$$

Proof.

- Notice that

$$\begin{aligned} A_j(\vec{b}) &= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix} \\ &= \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_{j-1} & A\vec{x} & A\vec{e}_{j+1} & \cdots & A\vec{e}_n \end{bmatrix} \\ &= A \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix} \\ &= A\, I_j(\vec{x}) \end{aligned}$$

where

$$I_j(\vec{x}) = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & x_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & x_{j-1} & & & \\ & & & x_j & & & \\ & & & x_{j+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & x_n & & & 1 \end{bmatrix}$$

Proof. (continued)

- Hence, by taking the determinants on both sides, we have

$$\begin{aligned} \det(A_j(\vec{b})) &= \det(A\, I_j(\vec{x})) \\ &= \det(A)\det(I_j(\vec{x})) \end{aligned}$$

- And because $\det(A) \neq 0$, we can then write:

$$\det(I_j(\vec{x})) = \frac{\det(A_j(\vec{b}))}{\det(A)}$$

- Finally, notice that $\det(I_j(\vec{x})) = \cdots = x_j$.

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Example

Solve for x_3:

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \quad \text{and} \quad \det A_3 = -6.$$

Therefore, $x_3 = \frac{-6}{-4} = \frac{3}{2}$.

Example (continued)

For practice, you should compute $\det A_1$ and $\det A_2$, where

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

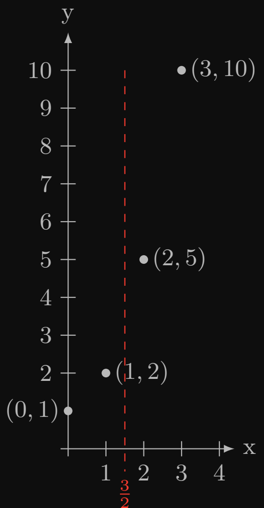
and then solve for x_1 and x_2.

Solution. $x_1 = -1$, $x_2 = \frac{7}{2}$.

Polynomial Interpolation and Vandermonde Determinant

Problem

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = 3/2$.



Solution

We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{aligned}
p(0) &= r_0 = 1 \\
p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\
p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\
p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10
\end{aligned}$$

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

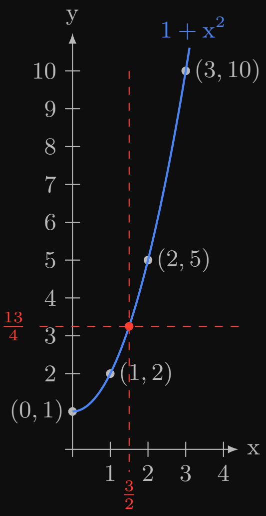
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

The estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$



y
$1 + x^2$
10
9
8
7
6
5
4
3
2
(3, 10)
(2, 5)
$\frac{13}{4}$
(1, 2)
(0, 1)
x
1
2
3
4
$\frac{3}{2}$

Theorem (Polynomial Interpolation)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The polynomial $p(x)$ is called the interpolating polynomial for the data.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$. $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$\begin{aligned}
r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\
r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\
r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\
\vdots \qquad\qquad & \vdots \quad \vdots \\
r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

The coefficient matrix for this system is

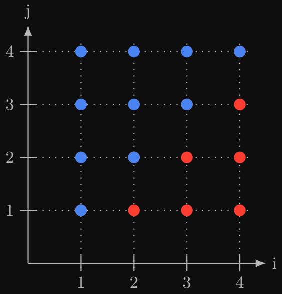
$$\begin{bmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_n & x_n^2 & \cdots & x_n^{n-1}
\end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

Theorem (Vandermonde Determinant)

Let $a_1, a_2, \ldots, a_n$ be real numbers, $n \geq 2$. The corresponding Vandermonde determinant is

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1\leq j<i\leq n} (a_i - a_j).$$



Proof.

We will prove this by induction. It is clear that when $n = 2$,

$$\det\begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1 = \prod_{1\le j<i\le 2}(a_i - a_j).$$

Assume that it is true for $n-1$. Now let's consider the case n. Denote

$$p(x) := \det\begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}.$$

Proof. (continued)

Because $p(a_1) = \cdots = p(a_{n-1}) = 0$ (why?), $p(x)$ has to take the following form:

$$p(x) = c(x - a_1)(x - a_2)\cdots(x - a_{n-1}).$$

To identify the constant c, notice that c is the coefficient for x^{n-1}. By cofactor expansion of the determinant along the last row,

$$\begin{aligned} c &= (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix} \\ &= \prod_{1 \le j < i \le n-1} (a_i - a_j). \end{aligned}$$

Proof. (continued)

Hence,

$$p(a_n) = \left(\prod_{1 \le j < i \le n-1} (a_i - a_j) \right) \times (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})$$

$$= \prod_{1 \le j < i \le n} (a_i - a_j).$$

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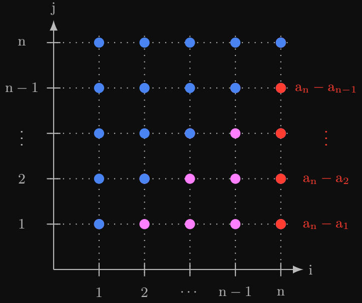

Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to the previous theorem, this determinant is equal to

$$\begin{aligned} & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ = \quad & (1 - 0)(2 - 0)(2 - 1)(3 - 0)(3 - 1)(3 - 2) = 2 \times 3 \times 2 \\ = \quad & 12. \end{aligned}$$

As a consequence of the theorem, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$