Math 221: LINEAR ALGEBRA

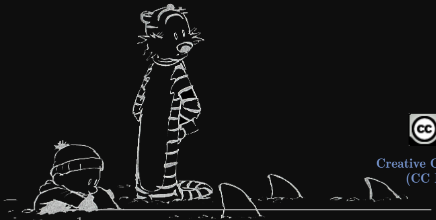
Chapter 3. Determinants and Diagonalization

§3-2. Determinants and Matrix Inverses

Le Chen[1]

Emory University, 2020 Fall

(last updated on 10/26/2020)



[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Determinants and Matrix Inverses

Adjugates

Cramer’s Rule

Polynomial Interpolation and Vandermonde Determinant

Determinants and Matrix Inverses

Determinants and Matrix Inverses

Theorem (Product Theorem)

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$

Determinants and Matrix Inverses

Theorem (Product Theorem)

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$

Theorem (Determinant of Matrix Inverse)

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}.$$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

$$= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c).$$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

$$= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c).$$

Therefore, A is invertible for all $c \neq 0, 3, -3$.

Theorem (Determinant of Matrix Transpose)

If A is an $n \times n$ matrix, then $\det(A^T) = \det A$.

Proof.

1. This is trivially true for all elementary matrices.
2. If A is not invertible, then neither is A^T (why?). Hence, $\det A = 0 = \det A^T$.
3. If A is invertible, then $A = E_k E_{k-1} \cdots E_2 E_1$. Hence, by case 1,

$$\det A = \cdots = \det A^T.$$

$\blacksquare$

Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

First,

$$\begin{aligned} \det(2A^{-1}) &= -4 \\ 2^3 \det(A^{-1}) &= -4 \\ \frac{1}{\det A} &= \frac{-4}{8} = -\frac{1}{2} \end{aligned}$$

Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

First,

$$\begin{aligned} \det(2A^{-1}) &= -4 \\ 2^3 \det(A^{-1}) &= -4 \\ \frac{1}{\det A} &= \frac{-4}{8} = -\frac{1}{2} \end{aligned}$$

Therefore, $\det A = -2$.

Example (continued)

Now,

$$\begin{aligned}
\det(A^3(B^{-1})^T) &= -4 \\
(\det A)^3 \det(B^{-1}) &= -4 \\
(-2)^3 \det(B^{-1}) &= -4 \\
(-8) \det(B^{-1}) &= -4 \\
\frac{1}{\det B} &= \frac{-4}{-8} = \frac{1}{2}
\end{aligned}$$

Example (continued)

Now,

$$\begin{aligned} \det(A^3(B^{-1})^T) &= -4 \\ (\det A)^3 \det(B^{-1}) &= -4 \\ (-2)^3 \det(B^{-1}) &= -4 \\ (-8)\det(B^{-1}) &= -4 \\ \frac{1}{\det B} &= \frac{-4}{-8} = \frac{1}{2} \end{aligned}$$

Therefore, $\det B = 2$.

Example

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \quad \text{and} \quad \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$.

Example

Suppose A, B and C are 4×4 matrices with

$$\det \mathrm{A} = -1, \det \mathrm{B} = 2, \quad \text{and} \quad \det \mathrm{C} = 1.$$

Find $\det(2\mathrm{A}^2(\mathrm{B}^{-1})(\mathrm{C}^{\mathrm{T}})^3\mathrm{B}(\mathrm{A}^{-1}))$.

$$\begin{aligned}
\det(2\mathrm{A}^2(\mathrm{B}^{-1})(\mathrm{C}^{\mathrm{T}})^3\mathrm{B}(\mathrm{A}^{-1})) &= 2^4(\det \mathrm{A})^2 \frac{1}{\det \mathrm{B}}(\det \mathrm{C})^3(\det \mathrm{B})\frac{1}{\det \mathrm{A}} \\
&= 16(\det \mathrm{A})(\det \mathrm{C})^3 \\
&= 16 \times (-1) \times 1^3 \\
&= -16.
\end{aligned}$$

Example

A square matrix A is orthogonal if and only if $A^T = A^{-1}$. What are the possible values of $\det A$ if A is orthogonal?

Example

A square matrix A is orthogonal if and only if $A^T = A^{-1}$. What are the possible values of $\det A$ if A is orthogonal?

Since $A^T = A^{-1}$,

$$\begin{aligned} \det A^T &= \det(A^{-1}) \\ \det A &= \frac{1}{\det A} \\ (\det A)^2 &= 1 \end{aligned}$$

Example

A square matrix A is orthogonal if and only if $A^T = A^{-1}$. What are the possible values of $\det A$ if A is orthogonal?

Since $A^T = A^{-1}$,

$$\begin{aligned} \det A^T &= \det(A^{-1}) \\ \det A &= \frac{1}{\det A} \\ (\det A)^2 &= 1 \end{aligned}$$

Assuming A is a real matrix, this implies that $\det A = \pm 1$, i.e., $\det A = 1$ or $\det A = -1$.

Adjugates

Adjugates

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of A defined as

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$\begin{aligned} A(\mathrm{adj}A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (\det A)I_2 \end{aligned}$$

Adjugates

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of A defined as

$$\text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$\begin{aligned} A(\text{adj}A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (\det A) I_2 \end{aligned}$$

Furthermore, if $\det A \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det A} \text{adj}A.$$

Definition

If A is an $n \times n$ matrix, then

$$\mathrm{adj} A = \left[\; c_{ij}(A) \; \right]^T,$$

where $c_{ij}(A)$ is the (i, j)-cofactor of A, i.e., $\mathrm{adj} A$ is the transpose of the cofactor matrix (matrix of cofactors).

Definition

If A is an $n \times n$ matrix, then

$$\operatorname{adj} A = \left[\ c_{ij}(A)\ \right]^T,$$

where $c_{ij}(A)$ is the (i,j)-cofactor of A, i.e., $\operatorname{adj} A$ is the transpose of the cofactor matrix (matrix of cofactors).

Reminder. $c_{ij}(A) = (-1)^{i+j} \det(A_{ij})$.

Example

Find adjA when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Example

Find adjA when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution.

$$\text{adj}A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$$

Example

Find $\mathrm{adj}A$ when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution.

$$\mathrm{adj}A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$$

Notice that

$$\begin{aligned} A(\mathrm{adj}A) &= \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix} \\ &= \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix} \end{aligned}$$

Example (continued)

Also,

$$\begin{aligned}
\det A &= \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 \\ 19 & 0 & 22 \\ 3 & 0 & -6 \end{vmatrix} \\
&= (-1)\begin{vmatrix} 19 & 22 \\ 3 & -6 \end{vmatrix} \\
&= 180,
\end{aligned}$$

Example (continued)

Also,

$$\begin{aligned}
\det A &= \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 & 3 \\ 19 & 0 & 22 \\ 3 & 0 & -6 \end{vmatrix} \\
&= (-1)\begin{vmatrix} 19 & 22 \\ 3 & -6 \end{vmatrix} \\
&= 180,
\end{aligned}$$

so in this example, we see that

$$A(\mathrm{adj}A) = (\det A)I$$

Theorem (The Adjugate Formula)

If A is an $n \times n$ matrix, then

$$A(\mathrm{adj}A) = (\det A)I = (\mathrm{adj}A)A.$$

Furthermore, if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}A.$$

Remark

Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Example

For an $n \times n$ matrix A, show that $\det(\mathrm{adj} A) = (\det A)^{n-1}$.

Example

For an $n \times n$ matrix A, show that $\det(\text{adj}A) = (\det A)^{n-1}$.

Using the adjugate formula,

$$\begin{aligned}
A(\text{adj}A) &= (\det A)I \\
\det(A(\text{adj}A)) &= \det((\det A)I) \\
(\det A) \times \det(\text{adj}A) &= (\det A)^n(\det I) \\
(\det A) \times \det(\text{adj}A) &= (\det A)^n
\end{aligned}$$

Example

For an $n \times n$ matrix A, show that $\det(\mathrm{adj}A) = (\det A)^{n-1}$.

Using the adjugate formula,

$$\begin{aligned} A(\mathrm{adj}A) &= (\det A)I \\ \det(A(\mathrm{adj}A)) &= \det((\det A)I) \\ (\det A) \times \det(\mathrm{adj}A) &= (\det A)^n(\det I) \\ (\det A) \times \det(\mathrm{adj}A) &= (\det A)^n \end{aligned}$$

If $\det A \neq 0$, then divide both sides of the last equation by $\det A$:

$$\det(\mathrm{adj}A) = (\det A)^{n-1}.$$

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj} A) = 0, \tag{$\star$}$$

Example (continued)

For the case $\det \mathrm{A} = 0$, we claim that

$$\det \mathrm{A} = 0 \quad \Rightarrow \quad \det(\mathrm{adj}\mathrm{A}) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj}\mathrm{A}) = 0 = 0^{\mathrm{n}-1} = (\det \mathrm{A})^{\mathrm{n}-1}.$$

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj} A) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj} A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of ($\star$))

We will prove ($\star$) by contradiction.

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj}A) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj}A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of ($\star$))

We will prove ($\star$) by contradiction. Indeed, if $\det A = 0$, then

$$A(\mathrm{adj}A) = (\det A)I = (0)I = O,$$

i.e., $A(\mathrm{adj}A)$ is the zero matrix.

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj}A) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj}A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of ($\star$))

We will prove ($\star$) by contradiction. Indeed, if $\det A = 0$, then

$$A(\mathrm{adj}A) = (\det A)I = (0)I = O,$$

i.e., $A(\mathrm{adj}A)$ is the zero matrix. If $\det(\mathrm{adj}A) \neq 0$, then $\mathrm{adj}A$ would be invertible, and $A(\mathrm{adj}A) = 0$ would imply $A = O$.

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj}A) = 0, \qquad (\star)$$

which implies that

$$\det(\mathrm{adj}A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of $(\star)$)

We will prove $(\star)$ by contradiction. Indeed, if $\det A = 0$, then

$$A(\mathrm{adj}A) = (\det A)I = (0)I = O,$$

i.e., $A(\mathrm{adj}A)$ is the zero matrix. If $\det(\mathrm{adj}A) \neq 0$, then $\mathrm{adj}A$ would be invertible, and $A(\mathrm{adj}A) = 0$ would imply $A = O$. However, if $A = O$, then $\mathrm{adj}A = 0$ and is not invertible, and thus has determinant equal to zero, i.e., $\det(\mathrm{adj}A) = 0$, (a contradiction!)

Example (continued)

For the case $\det A = 0$, we claim that

$$\det A = 0 \quad \Rightarrow \quad \det(\mathrm{adj}A) = 0, \tag{$\star$}$$

which implies that

$$\det(\mathrm{adj}A) = 0 = 0^{n-1} = (\det A)^{n-1}.$$

Proof. (of $(\star)$)

We will prove $(\star)$ by contradiction. Indeed, if $\det A = 0$, then

$$A(\mathrm{adj}A) = (\det A)I = (0)I = O,$$

i.e., $A(\mathrm{adj}A)$ is the zero matrix. If $\det(\mathrm{adj}A) \neq 0$, then $\mathrm{adj}A$ would be invertible, and $A(\mathrm{adj}A) = 0$ would imply $A = O$. However, if $A = O$, then $\mathrm{adj}A = 0$ and is not invertible, and thus has determinant equal to zero, i.e., $\det(\mathrm{adj}A) = 0$, (a contradiction!) Therefore, $\det(\mathrm{adj}A) = 0$, i.e., $(\star)$ is true.
$\blacksquare$

Problem

Let A and B be $n \times n$ matrices. Show that $\det(A + B^T) = \det(A^T + B)$.

Problem

Let A and B be $n \times n$ matrices. Show that $\det(A + B^T) = \det(A^T + B)$.

Solution

Notice that

$$(A + B^T)^T = A^T + (B^T)^T = A^T + B.$$

Since a matrix and it's transpose have the same determinant

$$\begin{aligned} \det(A + B^T) &= \det((A + B^T)^T) \\ &= \det(A^T + B). \end{aligned}$$

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

(a) If $\mathrm{adj}(A)$ exists, then A is invertible.

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

(a) If $\mathrm{adj}(A)$ exists, then A is invertible.

(c) If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

(a) If $\mathrm{adj}(A)$ exists, then A is invertible.

(c) If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.

Example

Prove or give a counterexample to the following statement:

$$\text{If } \det A = 1, \text{ then } \mathrm{adj}A = A.$$

Cramer's Rule

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $A\vec{x} = \vec{b}$ can be given in terms of determinants of matrices.

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $A\vec{x} = \vec{b}$ can be given in terms of determinants of matrices.

Theorem (Cramer's Rule)

Let A be an $n \times n$ invertible matrix, the solution to the system $A\vec{x} = \vec{b}$ of n equations in teh variables $x_1, x_2 \cdots x_n$ is given by

$$x_1 = \frac{\det\left(A_1(\vec{b})\right)}{\det A}, \quad x_2 = \frac{\det\left(A_2(\vec{b})\right)}{\det A}, \quad \cdots, \quad x_n = \frac{\det\left(A_n(\vec{b})\right)}{\det A}$$

where, for each j, the matrix $A_j(\vec{b})$ is obtained from A by replacing column j with $\vec{b}$:

$$A_j(\vec{b}) = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix}$$

Proof.

- Notice that

$$\begin{aligned} A_j(\vec{b}) &= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix} \\ &= \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_{j-1} & A\vec{x} & A\vec{e}_{j+1} & \cdots & A\vec{e}_n \end{bmatrix} \\ &= A \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix} \\ &= A\, I_j(\vec{x}) \end{aligned}$$

Proof.

- Notice that

$$
\begin{aligned}
A_j(\vec{b}) &= \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_{j-1} & \vec{b} & \vec{a}_{j+1} & \cdots & \vec{a}_n \end{bmatrix} \\
&= \begin{bmatrix} A\vec{e}_1 & \cdots & A\vec{e}_{j-1} & A\vec{x} & A\vec{e}_{j+1} & \cdots & A\vec{e}_n \end{bmatrix} \\
&= A \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix} \\
&= A\, I_j(\vec{x})
\end{aligned}
$$

where

$$
I_j(\vec{x}) = \begin{bmatrix} \vec{e}_1 & \cdots & \vec{e}_{j-1} & \vec{x} & \vec{e}_{j+1} & \cdots & \vec{e}_n \end{bmatrix}
= \begin{bmatrix}
1 & & & x_1 & & & \\
& \ddots & & \vdots & & & \\
& & 1 & x_{j-1} & & & \\
& & & x_j & & & \\
& & & x_{j+1} & 1 & & \\
& & & \vdots & & \ddots & \\
& & & x_n & & & 1
\end{bmatrix}
$$

Proof. (continued)

- Hence, by taking the determinants on both sides, we have

$$\begin{aligned} \det(A_j(\vec{b})) &= \det(A\, I_j(\vec{x})) \\ &= \det(A) \det(I_j(\vec{x})) \end{aligned}$$

- And because $\det(A) \neq 0$, we can then write:

$$\det(I_j(\vec{x})) = \frac{\det(A_j(\vec{b}))}{\det(A)}$$

- Finally, notice that $\quad \det(I_j(\vec{x})) = \cdots$

Proof. (continued)

- Hence, by taking the determinants on both sides, we have

$$\begin{aligned} \det(A_j(\vec{b})) &= \det(A\, I_j(\vec{x})) \\ &= \det(A)\det(I_j(\vec{x})) \end{aligned}$$

- And because $\det(A) \neq 0$, we can then write:

$$\det(I_j(\vec{x})) = \frac{\det(A_j(\vec{b}))}{\det(A)}$$

- Finally, notice that $\quad \det(I_j(\vec{x})) = \cdots = x_j.$

$\blacksquare$

Example

Solve for x_3:

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

Example

Solve for x_3:

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Example

Solve for x_3:

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \quad \text{and} \quad \det A_3 = -6.$$

Example

Solve for x_3:

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & - & x_3 & = & -1 \\ 5x_1 & + & 2x_2 & & & = & 2 \\ x_1 & + & x_2 & - & x_3 & = & 1 \end{array}$$

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \quad \text{and} \quad \det A_3 = -6.$$

Therefore, $x_3 = \frac{-6}{-4} = \frac{3}{2}$.

Example (continued)

For practice, you should compute $\det A_1$ and $\det A_2$, where

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

and then solve for x_1 and x_2.

Example (continued)

For practice, you should compute $\det A_1$ and $\det A_2$, where

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

and then solve for x_1 and x_2.

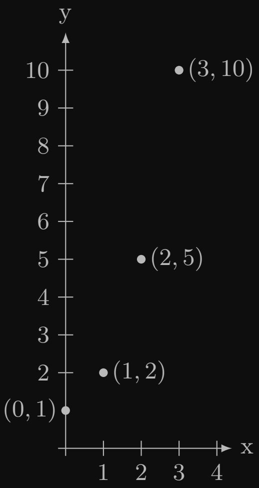
Solution. $x_1 = -1$, $x_2 = \frac{7}{2}$.

Polynomial Interpolation and Vandermonde Determinant

Polynomial Interpolation and Vandermonde Determinant

Problem

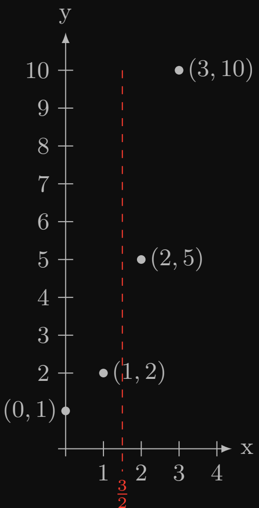
Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = 3/2$.



Polynomial Interpolation and Vandermonde Determinant

Problem

Given data points $(0,1)$, $(1,2)$, $(2,5)$ and $(3,10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = 3/2$.



Solution

We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{aligned}
p(0) &= r_0 = 1 \\
p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\
p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\
p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10
\end{aligned}$$

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

$$\left[\begin{array}{cccr|r} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

$$\left[\begin{array}{cccr|r} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

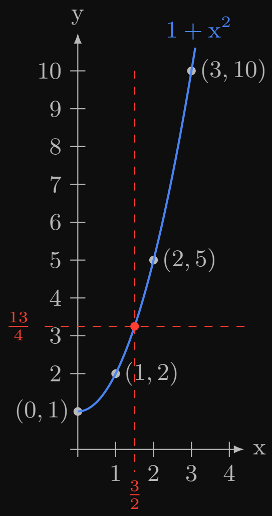
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

The estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$



y
$1 + x^2$
10
9
8
7
6
5
4
3
2
$\frac{13}{4}$
$(3, 10)$
$(2, 5)$
$(1, 2)$
$(0, 1)$
x
1
2
3
4
$\frac{3}{2}$

Theorem (Polynomial Interpolation)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

Theorem (Polynomial Interpolation)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The polynomial $p(x)$ is called the interpolating polynomial for the data.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$. $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$
\begin{aligned}
r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\
r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\
r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\
\vdots \qquad\qquad &\vdots \quad \vdots \\
r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n
\end{aligned}
$$

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$. $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$\begin{aligned}
r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\
r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\
r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\
\vdots \qquad &\vdots \quad \vdots \\
r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$. $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1}x^{n-1}$:

$$\begin{aligned}
r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\
r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\
r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\
\vdots \qquad &\vdots \quad \vdots \\
r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

The coefficient matrix for this system is

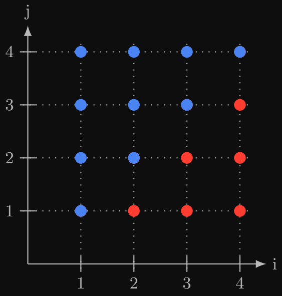
$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

Theorem (Vandermonde Determinant)

Let $a_1, a_2, \ldots, a_n$ be real numbers, $n \geq 2$. The corresponding Vandermonde determinant is

$$\det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} = \prod_{1 \leq j < i \leq n} (a_i - a_j).$$



Proof.

We will prove this by induction. It is clear that when $n = 2$,

$$\det \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1 = \prod_{1 \le j < i \le 2} (a_i - a_j).$$

Proof.

We will prove this by induction. It is clear that when $n = 2$,

$$\det\begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1 = \prod_{1\le j<i\le 2} (a_i - a_j).$$

Assume that it is true for $n - 1$.

Proof.

We will prove this by induction. It is clear that when $n = 2$,

$$\det \begin{pmatrix} 1 & a_1 \\ 1 & a_2 \end{pmatrix} = a_2 - a_1 = \prod_{1 \le j < i \le 2} (a_i - a_j).$$

Assume that it is true for $n - 1$. Now let's consider the case n. Denote

$$p(x) := \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \\ 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix}.$$

Proof. (continued)

Because $p(a_1) = \cdots = p(a_{n-1}) = 0$ (why?), $p(x)$ has to take the following form:

$$p(x) = c(x - a_1)(x - a_2)\cdots(x - a_{n-1}).$$

To identify the constant c, notice that c is the coefficient for x^{n-1}. By cofactor expansion of the determinant along the last row,

$$\begin{aligned} c &= (-1)^{n+n} \det \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-1} \end{bmatrix} \\ &= \prod_{1 \le j < i \le n-1} (a_i - a_j). \end{aligned}$$

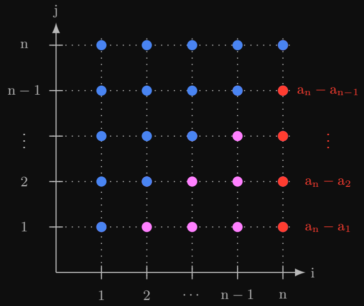
Proof. (continued)

Hence,

$$p(a_n) = \left(\prod_{1 \le j < i \le n-1} (a_i - a_j) \right) \times (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1})$$

$$= \prod_{1 \le j < i \le n} (a_i - a_j).$$

■



Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to the previous theorem, this determinant is equal to

$$\begin{aligned} & (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ = \; & (1-0)(2-0)(2-1)(3-0)(3-1)(3-2) = 2 \times 3 \times 2 \\ = \; & 12. \end{aligned}$$

As a consequence of the theorem, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

As a consequence of the theorem, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$