# Math 221: LINEAR ALGEBRA

## Chapter 4. Vector Geometry

## §4-2. Projections and Planes


Le Chen[1]

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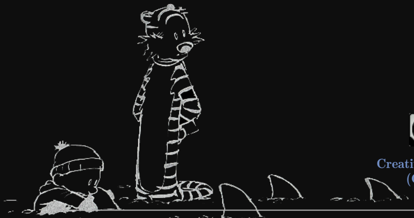




[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

The Dot Product and Angles

Projections

Planes

Cross Product

Shortest Distances

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

# The Dot Product and Angles

## Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \cdot \vec{v}$ is a scalar.

## Remark

Another way to think about the dot product is as the $1 \times 1$ matrix

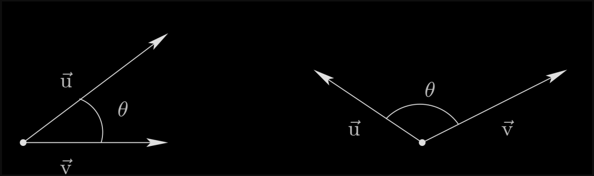
$$\vec{u}^T\vec{v} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 + y_1y_2 + z_1z_2 \end{bmatrix}.$$

## Theorem ( Properties of the Dot Product )

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \cdot \vec{v}$ is a real number.
2. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. (commutative property)
3. $\vec{u} \cdot \vec{0} = 0$.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$.
5. $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (k\vec{v})$. (associative property)
6. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$. (distributive properties)
   $\vec{u} \cdot (\vec{v} - \vec{w}) = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{w}$.

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$). There is a unique angle $\theta$ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.



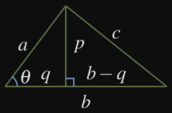

## Theorem

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let $\theta$ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

Proof.

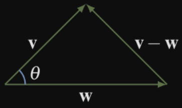
We first prove the Law of Cosines – a generalization of the Pythagorean theorem:




$$\begin{aligned} c^2 = p^2 + (b-q)^2 &= a^2 \sin^2\theta + (b - a\cos\theta)^2 \\ &= a^2\left(\sin^2\theta + \cos^2\theta\right) + b^2 - 2ab\cos\theta \\ &= a^2 + b^2 - 2ab\cos\theta. \end{aligned}$$

## Proof. (continued)

In terms of vectors, we see that




$$||\vec{v}-\vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}||\cos\theta$$

$$||$$

$$(\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) = ||\vec{v}||^2 - 2\vec{v}\cdot\vec{w} + ||\vec{w}||^2$$

$$\Downarrow$$

$$||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{v}||\,||\vec{w}||\cos\theta = ||\vec{v}||^2 - 2\vec{v}\cdot\vec{w} + ||\vec{w}||^2$$

$$\Downarrow$$

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta.$$

∎

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- If $0 \le \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \le \pi$, then $\cos\theta < 0$.

Therefore, for nonzero vectors $\vec{u}$ and $\vec{v}$,

- $\vec{u} \cdot \vec{v} > 0$ if and only if $0 \le \theta < \frac{\pi}{2}$.
- $\vec{u} \cdot \vec{v} = 0$ if and only if $\theta = \frac{\pi}{2}$.
- $\vec{u} \cdot \vec{v} < 0$ if and only if $\frac{\pi}{2} < \theta \le \pi$.

## Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

## Theorem

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \cdot \vec{v} = 0$.

## Problem

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

## Solution

$\vec{u} \cdot \vec{v} = 1$, $||\vec{u}|| = \sqrt{2}$ and $||\vec{v}|| = \sqrt{2}$.
Therefore,

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\ ||\vec{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, $\theta = \frac{\pi}{3}$.

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.

## Problem

Find the angle between $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

## Solution

$\vec{u} \cdot \vec{v} = 0$, and therefore the angle between the vectors is $\frac{\pi}{2}$.

## Problem

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both $\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

## Solution

There are infinitely many such vectors. Since $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$,

$$\begin{aligned} \vec{v} \cdot \vec{u} &= -x - 3y + 2z = 0 \\ \vec{v} \cdot \vec{w} &= y + z = 0 \end{aligned}$$

## Solution (continued)

This is a homogeneous system of two linear equation in three variables.

$$\left[\begin{array}{rrr|r} -1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Therefore, $\vec{v} = t\left[\begin{array}{r} 5 \\ -1 \\ 1 \end{array}\right]$ for all $t \in \mathbb{R}$.

## Problem

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

## Solution

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \quad \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

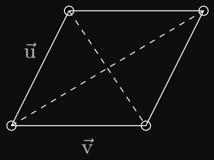
- $\overrightarrow{AB} \cdot \overrightarrow{AC} = 6 + 187 + 75 \neq 0.$
- $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\overrightarrow{AB}) \cdot \overrightarrow{BC} = -2 - 66 - 50 \neq 0.$
- $\overrightarrow{CA} \cdot \overrightarrow{CB} = (-\overrightarrow{AC}) \cdot (-\overrightarrow{BC}) = \overrightarrow{AC} \cdot \overrightarrow{BC} = 3 + 102 + 150 \neq 0.$

Because none of the angles is $\frac{\pi}{2}$, the triangle is not a right angle triangle.

## Problem

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

## Solution




Define the parallelogram (rhombus) by vectors $\vec{u}$ and $\vec{v}$.

Then the diagonals are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular.

$$\begin{aligned}(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) &= \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v}\\ &= ||\vec{u}||^2-\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{v}-||\vec{v}||^2\\ &= ||\vec{u}||^2-||\vec{v}||^2\\ &= 0, \qquad \text{since } ||\vec{u}||=||\vec{v}||.\end{aligned}$$

Therefore, the diagonals are perpendicular.

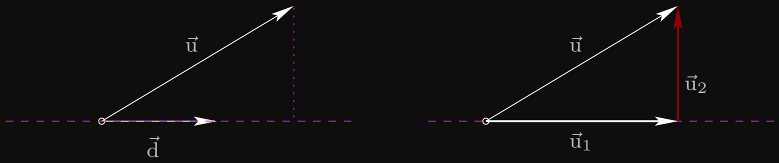
# Projections

Given two nonzero vectors $\vec{u}$ and $\vec{d}$, one can always express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.


$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \mathrm{proj}_{\vec{d}}\vec{u}$.

How to find $\vec{u}_1 = \mathrm{proj}_{\vec{d}}\vec{u}$ ?

$$
\begin{aligned}
\vec{u}_2 \cdot \vec{u}_1 &= 0 && (\vec{u}_1 \perp \vec{u}_2) \\
\vec{u}_2 \cdot (t\vec{d}) &= 0 && (\vec{u}_1 = t\vec{d}) \\
t(\vec{u}_2 \cdot \vec{d}) &= 0 \\
\vec{u}_2 \cdot \vec{d} &= 0 && (t \neq 0 \text{ b.c. } \vec{u} \neq \vec{0}) \\
(\vec{u} - \vec{u_1}) \cdot \vec{d} &= 0 && (\vec{u}_1 + \vec{u}_2 = \vec{u}) \\
\vec{u} \cdot \vec{d} - \vec{u_1} \cdot \vec{d} &= 0 \\
\vec{u} \cdot \vec{d} - (t\vec{d}) \cdot \vec{d} &= 0 && (\vec{u}_1 = t\vec{d}) \\
\vec{u} \cdot \vec{d} - t(\vec{d} \cdot \vec{d}) &= 0 \\
\vec{u} \cdot \vec{d} - t||\vec{d}||^2 &= 0 \\
\vec{u} \cdot \vec{d} &= t||\vec{d}||^2 \\
t &= \frac{\vec{u} \cdot \vec{d}}{||\vec{d}||^2} && (\vec{d} \neq \vec{0}) \\
\vec{u}_1 &= \frac{\vec{u} \cdot \vec{d}}{||\vec{d}||^2}\vec{d} && (\vec{u}_1 = t\vec{d})
\end{aligned}
$$




## Theorem

Let $\vec{u}$ and $\vec{d}$ be vectors with $\vec{d} \neq \vec{0}$.

1. The projection of $\vec{u}$ onto $\vec{d}$ is

$$\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u} = \frac{\vec{u}\cdot\vec{d}}{||\vec{d}||^2}\vec{d}.$$

2.

$$\vec{u}_2 = \vec{u} - \frac{\vec{u}\cdot\vec{d}}{||\vec{d}||^2}\vec{d}$$

is orthogonal to $\vec{d}$.

unit vector

$$\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u} = \left(\vec{u} \cdot \frac{\vec{d}}{||\vec{d}||}\right) \frac{\vec{d}}{||\vec{d}||} = \frac{\vec{u} \cdot \vec{d}}{||\vec{d}||^2}\vec{d}$$

length

direction

## Problem

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{u}_1$ and $\vec{u}_2$ so that $\vec{u} = \vec{u}_1 + \vec{u}_2$, with $\vec{u}_1$ parallel to $\vec{v}$ and $\vec{u}_2$ orthogonal to $\vec{v}$.

## Solution

$$\vec{u}_1 = \text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}\vec{v} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 \\ -16 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -16/11 \\ 5/11 \end{bmatrix}.$$

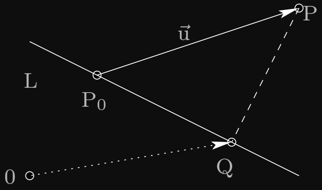
## Problem

Let $P(3, 2, -1)$ be a point in $\mathbb{R}^3$ and $L$ a line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Find the shortest distance from $P$ to $L$, and find the point $Q$ on $L$ that is closest to $P$.

## Solution




Let $P_0 = P_0(2, 1, 3)$ be a point on $L$, and let $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$. Then $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}} \overrightarrow{P_0P}$, $\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q}$, and the shortest distance from $P$ to $L$ is the length of $\overrightarrow{QP}$, where $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q}$.

## Solution (continued)

$\overrightarrow{P_0P} = \begin{bmatrix} 1 & 1 & -4 \end{bmatrix}^T$, $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P} \cdot \vec{d}}{||\vec{d}||^2}\vec{d} = \frac{10}{14}\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix}.$$

Therefore,

$$\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 29 \\ 2 \\ 11 \end{bmatrix},$$

so $Q = Q\left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7}\right)$.

## Solution (continued)

Finally, the shortest distance from $P(3, 2, -1)$ to $L$ is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

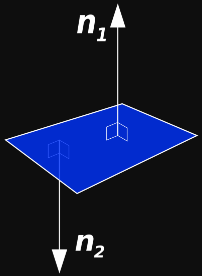
Therefore the shortest distance from $P$ to $L$ is

$$||\overrightarrow{QP}|| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$

# Planes

## Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \cdot \vec{v} = 0$ for every vector $\vec{v}$ in the plane.




Given a point $P_0$ and a nonzero vector $\vec{n}$, there is a unique plane containing $P_0$ and orthogonal to $\vec{n}$.

Consider a plane containing a point $P_0$ and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane.

Then

$$\vec{n} \cdot \overrightarrow{P_0P} = 0,$$

or, equivalently,

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0,$$

and is a vector equation of the plane.

$$\vec{n} \cdot (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0 \iff \vec{n} \cdot \overrightarrow{0P} = \vec{n} \cdot \overrightarrow{0P_0}$$

by setting $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, $\vec{n} = \begin{bmatrix} a & b & c \end{bmatrix}^T$

$$\iff \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$\iff ax + by + cz = ax_0 + by_0 + cz_0,$$

setting $d = ax_0 + by_0 + cz_0$ – a scalar

$$\iff \boxed{ax + by + cz = d}, \text{ where } a, b, c, d \in \mathbb{R}.$$

This is the scalar equation of the plane.

## Problem

Find an equation of the plane containing $P_0(1,-1,0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

## Solution

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

A scalar equation of this plane is

$$-3x + 5y + 2z = -3(1) + 5(-1) + 2(0) = -8,$$

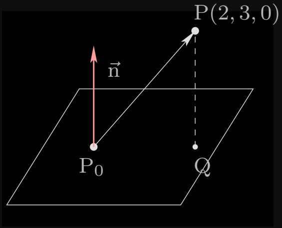
i.e., the plane has scalar equation

$$-3x + 5y + 2z = -8.$$

## Problem

Find the shortest distance from the point $P(2,3,0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

## Solution




Pick an arbitrary point $P_0$ on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$,

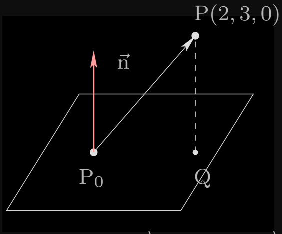
$||\overrightarrow{QP}||$ is the shortest distance,

and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$. Choose $P_0 = P_0(0,0,-1)$. Then

$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T$$

## Solution (continued)




$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \frac{\overrightarrow{P_0P}\cdot\vec{n}}{||\vec{n}||^2}\vec{n} = \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Since $||\overrightarrow{QP}|| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

To find Q, we have

$$\begin{aligned} \overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} &= \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T - \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T \\ &= \frac{1}{27}\begin{bmatrix} -16 & 67 & -14 \end{bmatrix}^T. \end{aligned}$$

Therefore $Q = Q\left(-\frac{16}{27}, \frac{67}{27}, -\frac{14}{27}\right)$.

## Remark

Here is a general answer: the distance from $P(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$ is

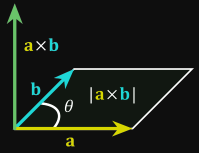
$$\text{distance} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

# The Cross Product

## Definition

Let $\vec{u} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$




## Remark

$\vec{u} \times \vec{v}$ is a vector:

- Direction: orthogonal to both $\vec{u}$ and $\vec{v}$.
- Size: the area of the corresponding parallelogram.

## Remark

A mnemonic device:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & x_1 & x_2 \\ \vec{j} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Or equivalently,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

## Theorem

Let $\vec{\mathrm{v}}, \vec{\mathrm{w}} \in \mathbb{R}^3$.

1. $\vec{\mathrm{v}} \times \vec{\mathrm{w}}$ is orthogonal to both $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$.
2. If $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ are both nonzero, then $\vec{\mathrm{u}} \times \vec{\mathrm{w}} = \vec{0}$ if and only if $\vec{\mathrm{v}}$ and $\vec{\mathrm{w}}$ are parallel.

## Problem

Find all vectors orthogonal to both $\vec{u} = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$. (We previously solved this using the dot product.)

## Solution

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{vmatrix} = -5\vec{i} + \vec{j} - \vec{k} = \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}.$$

Any scalar multiple of $\vec{u} \times \vec{v}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$, so

$$t \begin{bmatrix} -5 \\ 1 \\ -1 \end{bmatrix}, \quad \forall t \in \mathbb{R},$$

gives all vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

(Compare this with our earlier answer.)

## Problem

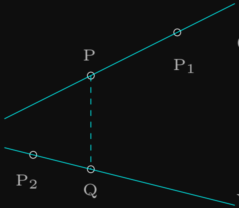
Given two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

A. Find the shortest distance between $L_1$ and $L_2$.

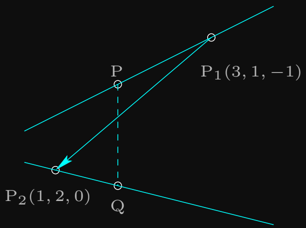
B. Find the points $P$ on $L_1$ and $Q$ on $L_2$ that are closest together.

## Solution




Choose $P_1(3, 1, -1)$ on $L_1$ and $P_2(1, 2, 0)$ on $L_2$.

Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ denote direction vectors for $L_1$ and $L_2$, respectively.




$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

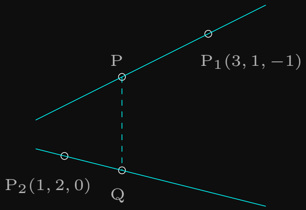
The shortest distance between $L_1$ and $L_2$ is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\overrightarrow{P_1P_2} = \frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{||\vec{n}||^2}\vec{n}, \quad \text{and} \quad ||\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}|| = \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{||\vec{n}||}.$$

Therefore, the shortest distance between $L_1$ and $L_2$ is $\frac{|-8|}{\sqrt{14}} = \frac{4}{7}\sqrt{14}$.

## Solution B.




$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both $L_1$ and $L_2$, so

$$\overrightarrow{PQ} \cdot \vec{d}_1 = 0 \quad \text{and} \quad \overrightarrow{PQ} \cdot \vec{d}_2 = 0,$$

i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

This system has unique solution $s = -\frac{5}{7}$ and $t = \frac{1}{7}$. Therefore,

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \quad \text{and} \quad Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right).$$

The shortest distance between $\mathrm{L}_1$ and $\mathrm{L}_2$ is $||\overrightarrow{\mathrm{PQ}}||$. Since

$$\mathrm{P} = \mathrm{P}\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \quad \text{and} \quad \mathrm{Q} = \mathrm{Q}\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{\mathrm{PQ}} = \frac{1}{7}\begin{bmatrix} 8 \\ 14 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 16 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix},$$

and

$$||\overrightarrow{\mathrm{PQ}}|| = \frac{1}{7}\sqrt{224} = \frac{4}{7}\sqrt{14}.$$

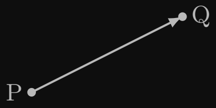
Therefore the shortest distance between $\mathrm{L}_1$ and $\mathrm{L}_2$ is $\frac{4}{7}\sqrt{14}$.

### Problem ( Challenge Problem )

Write yourself a plan to find the shortest distance in $\mathbb{R}^3$ between either a point, line or plane, to either a point, line or plane.

# Point-point distance

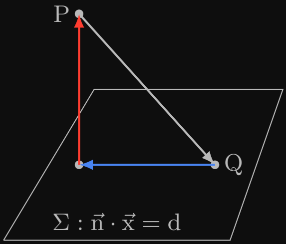
If P and Q are two points, then $\mathrm{d}\,(P, Q) = |\overrightarrow{PQ}|$.



# Point-plane distance

If $P$ is a point and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point $Q$, then

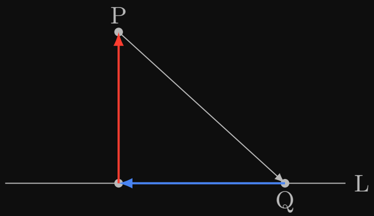
$$d(P, \Sigma) = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{|\vec{n}|}$$



# Point-line distance

If P is a point and L is a line $\vec{r}(t) = Q + t\vec{u}$, then

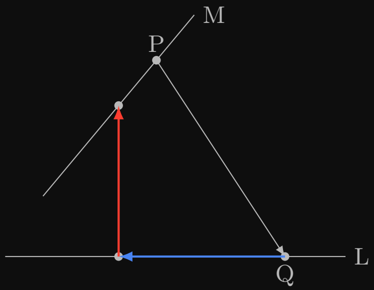
$$d(P, L) = \frac{\left|\overrightarrow{PQ} \times \vec{u}\right|}{|\vec{u}|}$$



# Line-line distance

If L is a line $\vec{r}(t) = Q + t\vec{u}$ and M is another line $\vec{s} = P + t\vec{v}$, then

$$d\,(L, M) = \frac{\left|\overrightarrow{PQ} \cdot (\vec{u} \times \vec{v})\right|}{|\vec{u} \times \vec{v}|}$$



# Plane-plane distance

If $\Sigma : \vec{n} \cdot \vec{x} = d$ and $\Theta : \vec{n} \cdot \vec{x} = e$ are two parallel planes, then

$$d(\Sigma, \Theta) = \frac{|e - d|}{|\vec{n}|}$$