Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry
§4-4. Linear Operators on $\mathbb{R}^3$

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Emory University, 2020 Fall

(last updated on 10/26/2020)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Rotations

Reflections

Multiple Actions

Summary

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Definition (Rotations in $\mathbb{R}^2$)

The transformation

$$\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$$

denotes counterclockwise rotation about the origin through an angle of θ.

Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$R_\theta(E_1) = R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

and

$$R_\theta(E_2) = R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

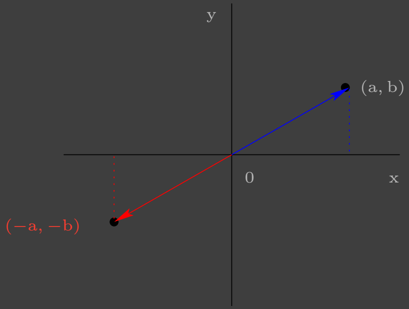
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π.



We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes a counterclockwise rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

Solution

First,

$$R_{\frac{\pi}{2}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

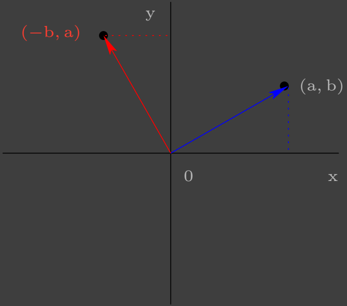
$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Rotation through $\pi/2$)

We denote by

$$R_{\pi/2} : \mathbb{R}^2 \to \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of $\pi/2$.



We see that $R_{\pi/2} \left[\begin{array}{c} a \\ b \end{array} \right] = \left[\begin{array}{c} -b \\ a \end{array} \right] = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \left[\begin{array}{c} a \\ b \end{array} \right]$, so $R_{\pi/2}$ is a matrix transformation.

Reflections

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

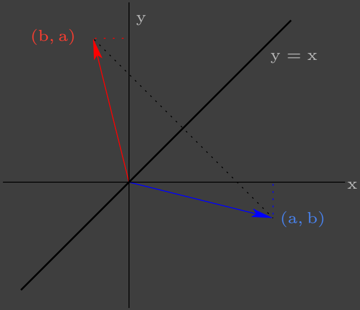
Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example

Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.

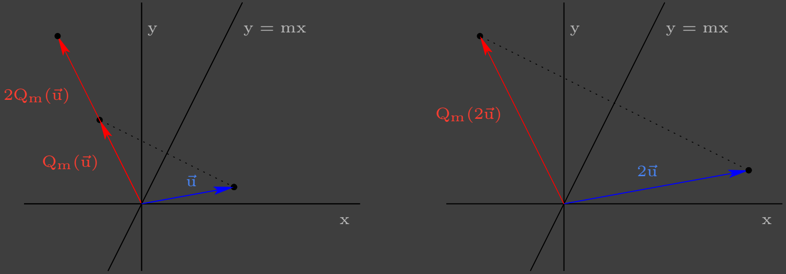


This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.



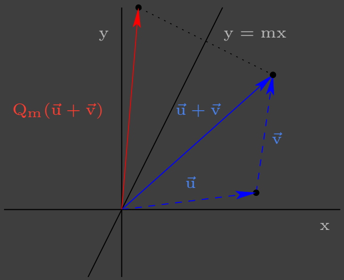
The figure indicates that $Q_m(2\vec{u}) = 2Q_m(\vec{u})$. In general, for any scalar k,

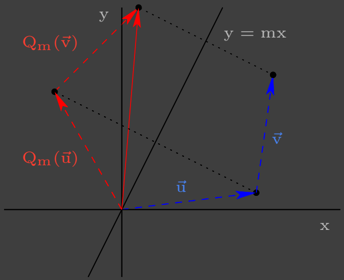
$$Q_m(k\vec{x}) = kQ_m(\vec{x}),$$

i.e., Q_m preserves scalar multiplication.

Example (Reflection in $y = mx$ preserves vector addition)

Let $\vec{u}, \vec{v} \in \mathbb{R}^2$.





The figure indicates that

$$Q_m(\vec{u}) + Q_m(\vec{v}) = Q_m(\vec{u} + \vec{v}),$$

i.e., Q_m preserves vector addition.

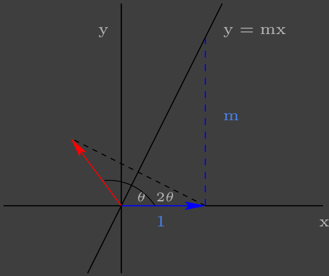
Q_m is a linear transformation

Since Q_m preserves addition and scalar multiplication, Q_m is a linear transformation, and hence a matrix transformation.

The matrix that induces Q_m can be found by computing $Q_m(E_1)$ and $Q_m(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

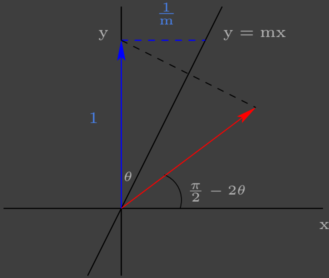
Example ($Q_m(E_1)$)



$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(E_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 \\ 2m \end{bmatrix}$$

Example ($Q_m(E_2)$)



$$\cos\theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{1+m^2}}$$

$$\begin{aligned} Q_m(E_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2}\cos(2\theta) + \sin\frac{\pi}{2}\sin(2\theta) \\ \sin\frac{\pi}{2}\cos(2\theta) - \cos\frac{\pi}{2}\sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} = \begin{bmatrix} 2\sin\theta\cos\theta \\ \cos^2\theta - \sin^2\theta \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 2m \\ m^2 - 1 \end{bmatrix} \end{aligned}$$

The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Multiple Actions

Problem

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Notice that $BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection matrix.

How do we know this?

Solution (continued)

Compare BA to

$$Q_m = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Now, since $1-m^2=0$, we know that $m=1$ or $m=-1$. But $\frac{2m}{1+m^2}=1>0$, so $m>0$, implying $m=1$.

Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y=x$. ∎

Problem (Relection followed by reflection)

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y-axis.

Solution

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and Q_Y is induced by

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA.

Solution (continued)

$$\mathrm{BA} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

What transformation does BA induce?

Rotation through an angle θ such that

$$\cos\theta = 0 \quad \text{and} \quad \sin\theta = -1.$$

Therefore, $\mathrm{Q_Y} \circ \mathrm{Q}_{-1} = \mathrm{R}_{-\frac{\pi}{2}} = \mathrm{R}_{\frac{3\pi}{2}}$.

Summary

In general,

- The composite of two rotations is a rotation

$$R_\theta \circ R_\eta = R_{\theta+\eta}.$$

- The composite of two reflections is a rotation.

$$Q_m \circ Q_n = R_\theta$$

where θ is $2\times$ the angle between lines $y = mx$ and $y = nx$.

- The composite of a reflection and a rotation is a reflection.

$$R_\theta \circ Q_n = Q_m \circ Q_n \circ Q_n = Q_m$$