

Math 221: LINEAR ALGEBRA

Chapter 4. Vector Geometry

§4-4. Linear Operators on $\mathbb{R}^3$

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Rotations

Reflections

Multiple Actions

Summary

NOTE: Much of this chapter is what you would learn in Multivariable Calculus.

You might find it interesting/useful to read.

But I will only cover the material important to this course.

Rotations

Definition

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T(\vec{x}) = A\vec{x} \text{ for each } \vec{x} \in \mathbb{R}^n$$

is called the **matrix transformation induced by A** .

Definition (Rotations in $\mathbb{R}^2$)

The transformation

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Rotation through an angle of θ preserves scalar multiplication.

Rotation through an angle of θ preserves vector addition.

R_θ is a linear transformation

Since R_θ preserves addition and scalar multiplication, R_θ is a linear transformation, and hence a matrix transformation.

The matrix that induces R_θ can be found by computing $R_\theta(E_1)$ and $R_\theta(E_2)$, where

$$E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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The Matrix for R_θ

The rotation $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear transformation, and is induced by the matrix

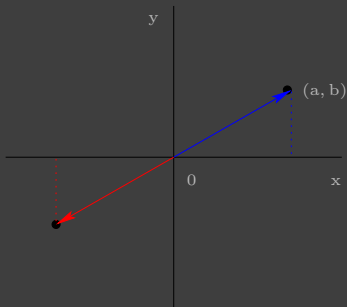
$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Example (Rotation through π)

We denote by

$$R_\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

counterclockwise rotation about the origin through an angle of π .

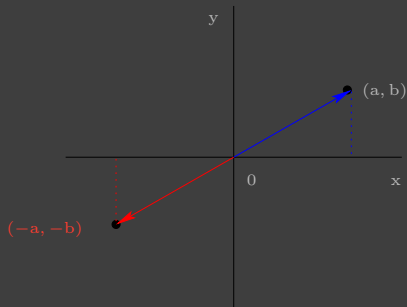


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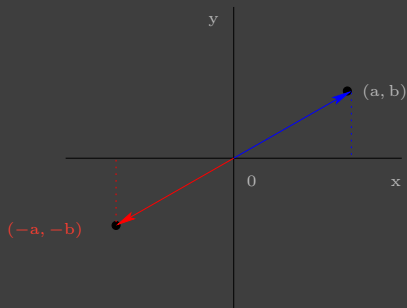


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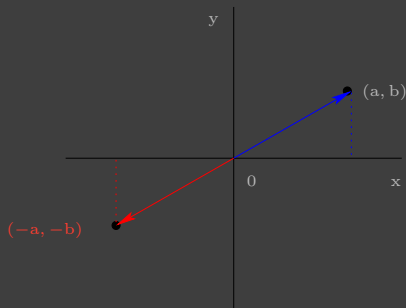
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We see that $R_\pi \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$, so R_π is a matrix transformation.

Problem

The transformation $R_{\frac{\pi}{2}} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denotes a **counterclockwise** rotation about the origin through an angle of $\frac{\pi}{2}$ radians. Find the matrix of $R_{\frac{\pi}{2}}$.

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Solution

First,

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Furthermore $R_{\frac{\pi}{2}}$ is a matrix transformation, and the matrix it is induced by is

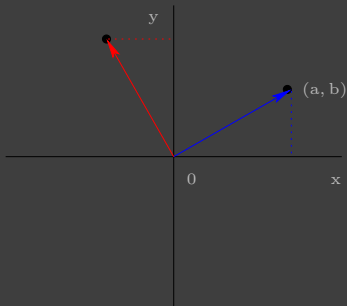
$$\begin{bmatrix} -b \\ a \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Example (Rotation through $\pi/2$)

We denote by

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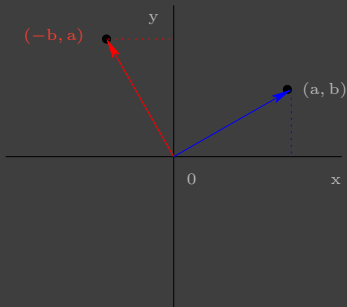


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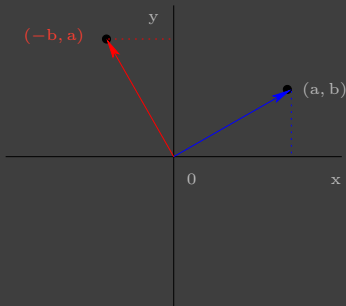


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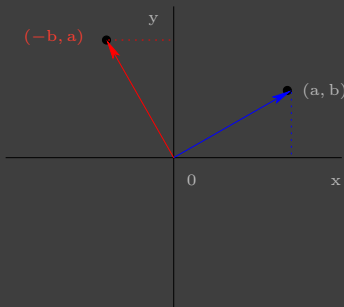
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Reflections

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

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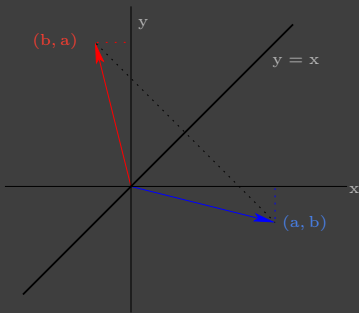
Example

In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$. This is a matrix transformation because

$$\begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

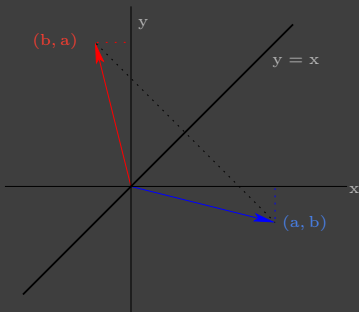
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Reflection in the line $y = x$ transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} b \\ a \end{bmatrix}$.



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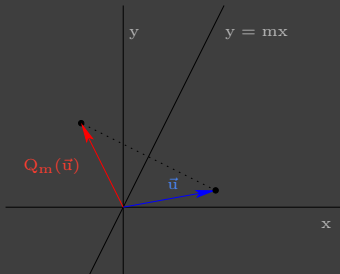


This is a matrix transformation because

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

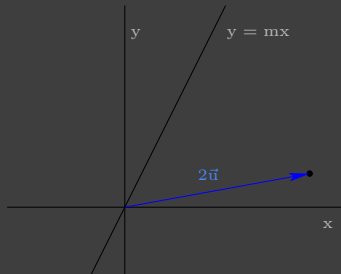
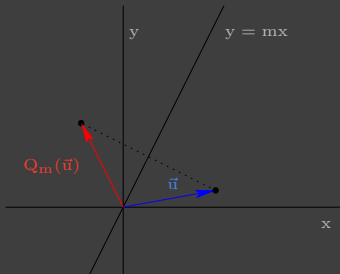
Example (Reflection in $y = mx$ preserves scalar multiplication)

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection in the line $y = mx$, and let $\vec{u} \in \mathbb{R}^2$.



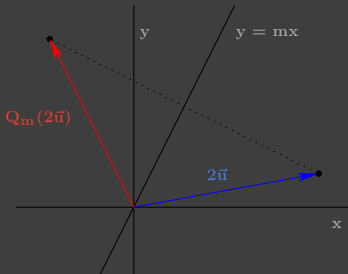
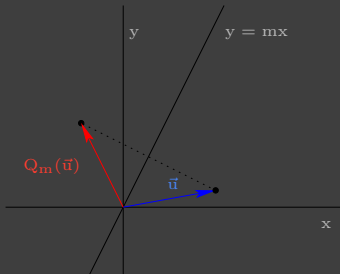
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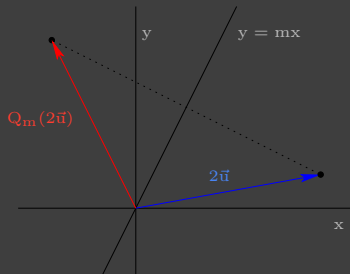
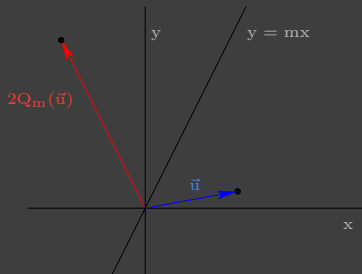
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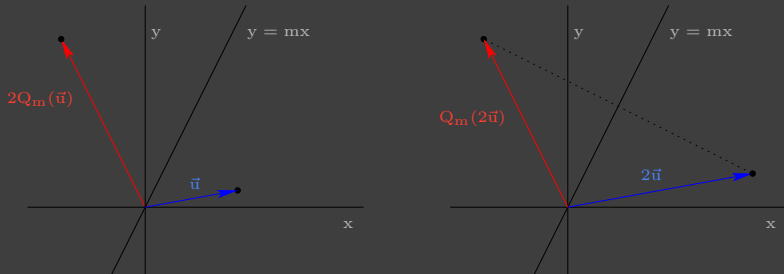
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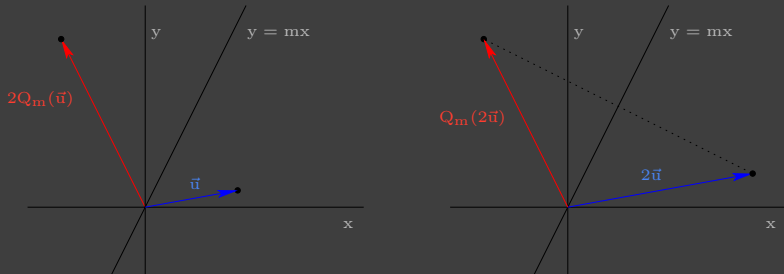


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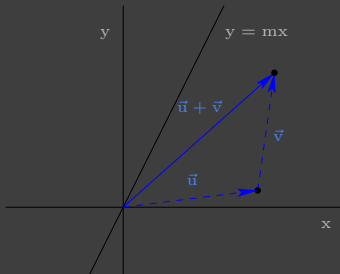
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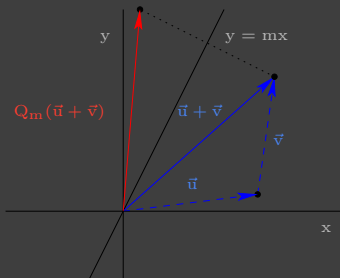
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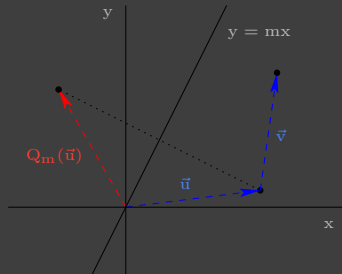
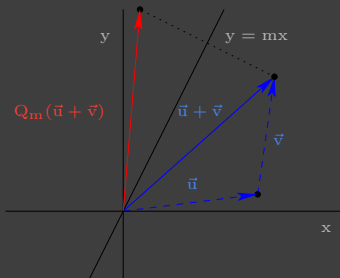
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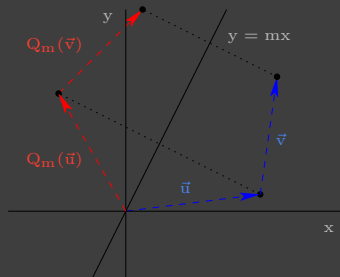
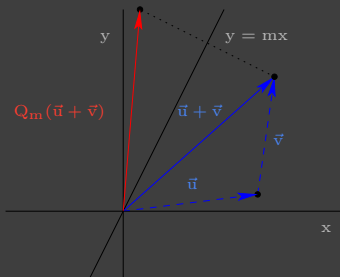
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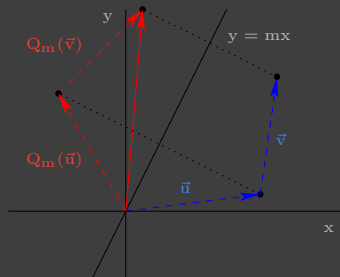
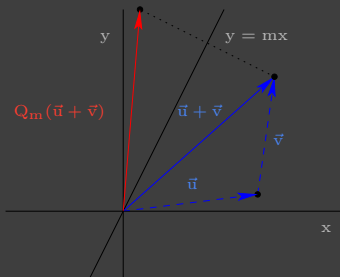
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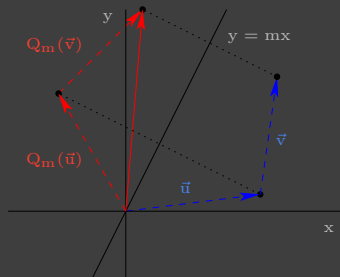
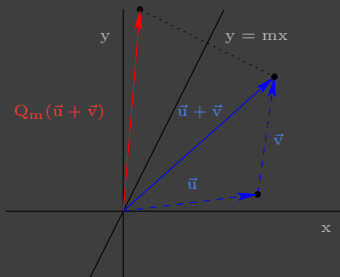
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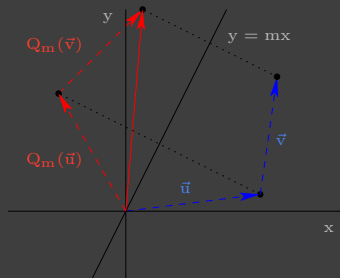
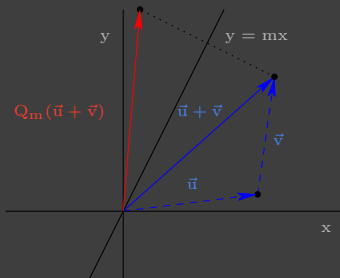


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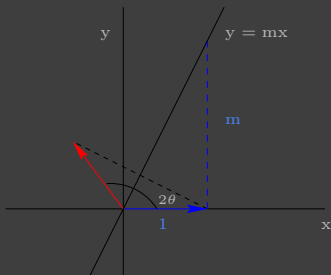
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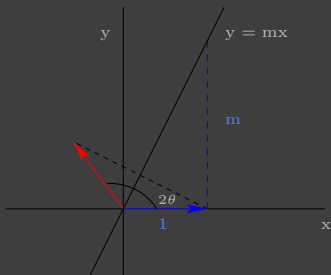
Example ($Q_m(E_1)$)



$$\cos \theta = \frac{1}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{m}{\sqrt{1+m^2}}$$

$$Q_m(E_1) = \begin{bmatrix} \cos(2\theta) \\ \sin(2\theta) \end{bmatrix}$$

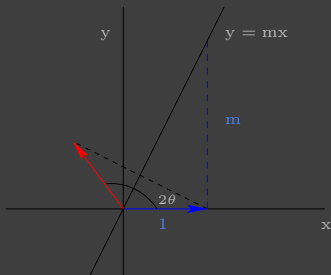
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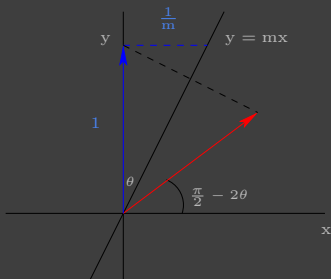
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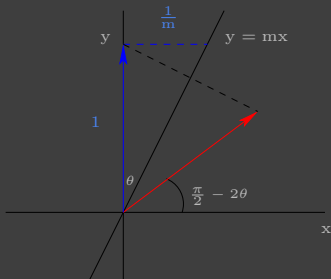
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$$\cos \theta = \frac{m}{\sqrt{1+m^2}} \quad \text{and} \quad \sin \theta = \frac{1}{\sqrt{1+m^2}}$$

$$Q_m(E_2) = \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{2} \cos(2\theta) + \sin \frac{\pi}{2} \sin(2\theta) \\ \sin \frac{\pi}{2} \cos(2\theta) - \cos \frac{\pi}{2} \sin(2\theta) \end{bmatrix}$$

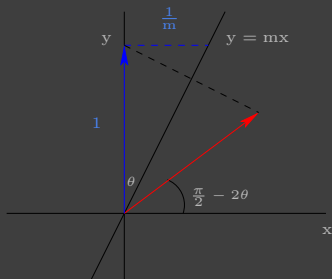
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$$\begin{aligned} Q_m(E_2) &= \begin{bmatrix} \cos(\frac{\pi}{2} - 2\theta) \\ \sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos \frac{\pi}{2} \cos(2\theta) + \sin \frac{\pi}{2} \sin(2\theta) \\ \sin \frac{\pi}{2} \cos(2\theta) - \cos \frac{\pi}{2} \sin(2\theta) \end{bmatrix} \\ &= \begin{bmatrix} \sin(2\theta) \\ \cos(2\theta) \end{bmatrix} \end{aligned}$$

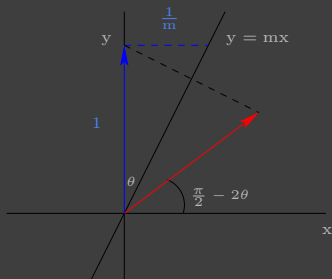
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The Matrix for Reflection in $y = mx$

The transformation $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, reflection in the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

Multiple Actions

Problem

Find the rotation or reflection that equals reflection in the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

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Solution

Let Q_0 denote the reflection in the x-axis, and $R_{\frac{\pi}{2}}$ denote the rotation through an angle of $\frac{\pi}{2}$. We want to find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution

Hence $R_{\frac{\pi}{2}} \circ Q_0$ is induced by

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How do we know this?

Solution (continued)

Compare BA to

$$Q_m = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$



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Therefore,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection in the line $y=x$. 

Problem (Reflection followed by reflection)

Find the rotation or reflection that equals reflection in the line $y = -x$ followed by reflection in the y -axis.

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Therefore, $Q_Y \circ Q_{-1}$ is induced by BA .

Solution (continued)

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What transformation does $\mathbf{BA}$ induce?

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Rotation through an angle θ such that

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Therefore, $\mathbf{Q}_Y \circ \mathbf{Q}_{-1} = \mathbf{R}_{-\frac{\pi}{2}} = \mathbf{R}_{\frac{3\pi}{2}}$.

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