Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-2. Independence and Dimension

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Independence

Geometric Examples

Independence, spanning, and matrices

Bases and Dimension

Finding Bases and Dimension

Linear Independence

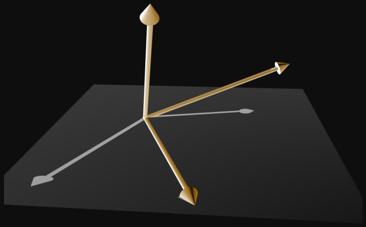
Linear Independence

Definition

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ be a subset of $\mathbb{R}^n$. The set S is linearly independent (or simply independent) if the following condition is satisfied:

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}_n \quad \Rightarrow \quad t_1 = t_2 = \cdots = t_k = 0$$

i.e., the only linear combination of vectors of S that vanishes (is equal to the zero vector) is the trivial one (all coefficients equal to zero).
A set that is not linearly independent is called dependent.



$\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k\}$		$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}_n$
Linearly Independent	$\Longleftrightarrow$	Trivial Solution
Linearly Dependent	$\Longleftrightarrow$	Nontrivial Solution

Example

Is $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ linearly independent?

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Suppose that a linear combination of these vectors vanishes, i.e., there exist $a, b, c \in \mathbb{R}$ so that

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Example (continued)

Solve the homogeneous system of three equation in three variables:

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The system has solutions $a = -2r, b = -3r, c = r$ for $r \in \mathbb{R}$, so it has nontrivial solutions. Therefore S is dependent.

Example (continued)

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The system has solutions $a = -2r, b = -3r, c = r$ for $r \in \mathbb{R}$, so it has nontrivial solutions. Therefore S is dependent. In particular, when $r = 1$ we see that

$$-2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., this is a nontrivial linear combination that vanishes.

Example

Consider the set $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\} \subseteq \mathbb{R}^n$, and suppose $t_1, t_2, \dots, t_n \in \mathbb{R}$ are such that

$$t_1\vec{e}_1 + t_2\vec{e}_2 + \cdots t_n\vec{e}_n = \vec{0}_n.$$

Since

$$t_1\vec{e}_1 + t_2\vec{e}_2 + \cdots t_n\vec{e}_n = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix},$$

the only linear combination that vanishes is the trivial one, i.e., the one with $t_1 = t_2 = \cdots = t_n = 0$. Therefore, $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is linearly independent.

Problem

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent subset of $\mathbb{R}^n$. Is $\{u \vec{+} v, 2u \vec{+} w, \vec{v} - 5\vec{w}\}$ linearly independent?

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Solution

In order to show the $\{u \vec{+} v, 2u \vec{+} w, \vec{v} - 5\vec{w}\}$ is linearly independent, we need to show that

$$a(\vec{u} + \vec{v}) + b(2\vec{u} + \vec{w}) + c(\vec{v} - 5\vec{w}) = \vec{0}_n \quad \Rightarrow \quad a = b = c = 0.$$

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$$(a + 2b)\vec{u} + (a + c)\vec{v} + (b - 5c)\vec{w} = \vec{0}_n.$$

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because $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent $\Downarrow$

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Let $\mathrm{X} \subseteq \mathbb{R}^{\mathrm{n}}$ and suppose that $\vec{0}_{\mathrm{n}} \in \mathrm{X}$. Show that X linearly dependent.

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Solution

Let $X = \{\vec{x_1}, \vec{x_2}, \ldots, \vec{x_k}\}$ for some $k \geq 1$, and suppose $\vec{x}_1 = \vec{0_n}$. Then

$$1\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_k = 1\vec{0} + 0\vec{x}_2 + \cdots + 0\vec{x}_k = \vec{0},$$

i.e., we have found a nontrivial linear combination of the vectors of X that vanishes. Therefore, X is dependent. ■

Example

Let $\vec{u} \in \mathbb{R}^n$ and let $S = \{\vec{u}\}$.

1. If $\vec{u} = \vec{0}_n$, then S is dependent (see the previous Problem).
2. If $\vec{u} \neq \vec{0}_n$, then S is independent: if $t\vec{u} = \vec{0}_n$ for some $t \in \mathbb{R}$, then $t = 0$.

As a consequence,

$$S = \{\vec{u}\} \text{ is independent} \qquad \Longleftrightarrow \qquad \vec{u} \neq \vec{0}_n$$

Example

$$A = \begin{bmatrix} 0 & 1 & -1 & 2 & 5 & 1 \\ 0 & 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is a row-echelon matrix.

Example

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$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix},$$

and suppose that $a\vec{u}_1 + b\vec{u}_2 + c\vec{u}_3 = \vec{0}_6$ for some $a, b, c \in \mathbb{R}$.

Example (continued)

This results in a system of six equations in three variables, whose augmented matrix is

$$\left[\begin{array}{rrr|r} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

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The solution to the system is easily determined to be $a = b = c = 0$, so the set $\{\vec{u}_1, \vec{u_2}, \vec{u}_3\}$ is independent.

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Remark

In general, the nonzero rows of any row-echelon matrix form an independent set of (row) vectors.

Theorem

Let $U = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \text{span}(U)$ has a unique representation as a linear combination of vectors of U.

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Proof.

Suppose that there is a vector $\vec{x} \in \text{span}(U)$ such that

$$\begin{aligned}
\vec{x} &= s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\
\vec{x} &= t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}.
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$$\Downarrow$$

$$\begin{aligned} \vec{0}_n = \vec{x} - \vec{x} &= (s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_k\vec{v}_k) - (t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k) \\ &= (s_1 - t_1)\vec{v}_1 + (s_2 - t_2)\vec{v}_2 + \cdots + (s_k - t_k)\vec{v}_k. \end{aligned}$$

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U is independent $\quad \Downarrow$

$$s_1 - t_1 = 0, \quad s_2 - t_2 = 0, \quad \cdots, s_k - t_k = 0$$

Theorem

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$$\text{U is independent} \quad \Downarrow$$

$$s_1 - t_1 = 0, \quad s_2 - t_2 = 0, \quad \cdots, s_k - t_k = 0$$

$$\Updownarrow$$

$$s_1 = t_1, \quad s_2 = t_2, \quad \cdots, s_k = t_k.$$

■

Two Geometric Examples

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Suppose that $\vec{u}$ and $\vec{v}$ are nonzero vectors in $\mathbb{R}^3$. Prove that $\{\vec{u}, \vec{v}\}$ is dependent if and only if $\vec{u}$ and $\vec{v}$ are parallel.

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Solution

($\Rightarrow$) If $\{\vec{u}, \vec{v}\}$ is dependent, then there exist $a, b \in \mathbb{R}$ so that $a\vec{u} + b\vec{v} = \vec{0}_3$ with a and b not both zero. By symmetry, we may assume that $a \neq 0$. Then $\vec{u} = -\frac{b}{a}\vec{v}$, so $\vec{u}$ and $\vec{v}$ are scalar multiples of each other, i.e., $\vec{u}$ and $\vec{v}$ are parallel.

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($\Leftarrow$) Conversely, if $\vec{u}$ and $\vec{v}$ are parallel, then there exists a $t \in \mathbb{R}$, $t \neq 0$, so that $\vec{u} = t\vec{v}$. Thus $\vec{u} - t\vec{v} = \vec{0}_3$, so we have a nontrivial linear combination of $\vec{u}$ and $\vec{v}$ that vanishes. Therefore, $\{\vec{u}, \vec{v}\}$ is dependent. ∎

Problem

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \mathrm{span}\{\vec{v}, \vec{w}\}$.

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Solution

($\Rightarrow$) If $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

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($\Rightarrow$) If $\vec{u} \in \mathrm{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

($\Leftarrow$) Now suppose that $\vec{u} \notin \mathrm{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \mathrm{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, implying that $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$, i.e., the only linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$ that vanishes is the trivial one. Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent. ■

Independence, spanning, and matrices

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Theorem

Suppose A is an $m \times n$ matrix with columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n \in \mathbb{R}^m$. Then

1. $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is independent if and only if $A\vec{x} = \vec{0}_m$ with $\vec{x} \in \mathbb{R}^n$ implies $\vec{x} = \vec{0}_n$.
2. $\mathbb{R}^m = \text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ if and only if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$.

Problem

Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$.

1. Are $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ linearly independent?
2. Do $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k$ span $\mathbb{R}^n$?

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Solution

To answer both question, simply let A be a matrix whose columns are the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$. Find R, a row-echelon form of A.

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1. "yes" if and only if each column of R has a leading one.

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Solution

To answer both question, simply let A be a matrix whose columns are the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^n$. Find R, a row-echelon form of A.

1. "yes" if and only if each column of R has a leading one.
2. "yes" if and only if each row of R has a leading one.

Problem (first seen earlier)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Problem (first seen earlier)

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Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$.

Problem (first seen earlier)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$. Apply row operations to get R, a row-echelon form of A:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix}$. Apply row operations to get R, a row-echelon form of A:

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last row of R consists only of zeros, $R\vec{x} = \vec{e}_4$ has no solution $\vec{x} \in \mathbb{R}^4$, implying that there is a $\vec{b} \in \mathbb{R}^4$ so that $A\vec{x} = \vec{b}$ has no solution $\vec{x} \in \mathbb{R}^4$. By previous Theorem, $\mathbb{R}^4 \neq \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. ■

Theorem

Let A be an $n \times n$ matrix. The following are equivalent.

1. A is invertible.
2. The columns of A are independent.
3. The columns of A span $\mathbb{R}^n$.
4. The rows of A are independent, i.e., the columns of A^T are independent.
5. The rows of A span the set of all $1 \times n$ rows, i.e., the columns of A^T span $\mathbb{R}^n$.

Problem (revisited)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Problem (revisited)

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$.

Show that $\text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\} \neq \mathbb{R}^4$.

Solution

Let $A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

By the previous Theorem, the columns of A span $\mathbb{R}^4$ if and only if A is invertible. Since $\det(A) = 0$ (row 2 is (-1) times row 1), A is not invertible, and thus $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ does not span $\mathbb{R}^4$. ■

Problem

Let

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}.$$

Is $\{\vec{u}, \vec{v}, \vec{w}\}$ independent?

Problem

Let

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}.$$

Is $\{\vec{u}, \vec{v}, \vec{w}\}$ independent?

Solution

Let $A = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$. From the previous Theorem, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if A is invertible.

Since

$$\det(A) = \det \begin{bmatrix} 1 & 3 & 3 \\ -1 & 2 & 5 \\ 0 & -1 & -2 \end{bmatrix} = -2,$$

and $-2 \neq 0$, A is invertible, and therefore $\{\vec{u}, \vec{v}, \vec{w}\}$ is an independent subset of $\mathbb{R}^3$. ■

Remark

Notice that $\{\vec{u}, \vec{v}, \vec{w}\}$ also spans $\mathbb{R}^3$.

Bases and Dimension

Theorem (Fundamental Theorem)

Let U be a subspace of $\mathbb{R}^n$ that is spanned by m vectors. If U contains a subset of k linearly independent vectors, then $k \leq m$.

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Definition

Let U be a subspace of $\mathbb{R}^n$. A set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is a basis of U if

1. $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ is linearly independent;
2. $U = \mathrm{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$.

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1. $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ is linearly independent;
2. $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$.

As a consequence of all this, if $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ is a basis of a subspace U, then every $\vec{u} \in U$ has a unique representation as a linear combination of the vectors $\vec{x}_i$, $1 \leq i \leq m$.

Example

The subset $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis of $\mathbb{R}^n$. (We've already seen that $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is linearly independent and that $\mathbb{R}^n = \text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$.)

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Example

In a previous problem, we saw that $\mathbb{R}^4 = \mathrm{span}(S)$ where

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

S is also linearly independent (prove this). Therefore, S is a basis of $\mathbb{R}^4$.

Theorem (Invariance Theorem)

If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ and $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Theorem (Invariance Theorem)

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Proof.

Let $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ and $T = \{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$. Since S spans U and T is independent, it follows from the Fundamental Theorem that $k \leq m$. Also, since T spans U and S is independent, it follows from the Fundamental Theorem that $m \leq k$. Since $k \leq m$ and $m \leq k$, $k = m$. ■

Theorem (Invariance Theorem)

If $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ and $\{\vec{y}_1, \vec{y}_2, \dots, \vec{y}_k\}$ are bases of a subspace U of $\mathbb{R}^n$, then $m = k$.

Proof.

Let $S = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ and $T = \{\vec{y}_1, \vec{y}_2, \dots, \vec{y}_k\}$. Since S spans U and T is independent, it follows from the Fundamental Theorem that $k \le m$. Also, since T spans U and S is independent, it follows from the Fundamental Theorem that $m \le k$. Since $k \le m$ and $m \le k$, $k = m$. ■

Definition

The dimension of a subspace U of $\mathbb{R}^n$ is the number of vectors in any basis of U, and is denoted $\dim(U)$.

Problem

In $\mathbb{R}^n$, what is the dimension of the subspace $\{\vec{0}_n\}$?

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The only basis of the zero subspace is the empty set, $\emptyset$:
(i) the empty set is (trivially) independent, and
(ii) any linear combination of no vectors is the zero vector.
Therefore, the zero subspace has dimension zero.

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The only basis of the zero subspace is the empty set, $\emptyset$:
(i) the empty set is (trivially) independent, and
(ii) any linear combination of no vectors is the zero vector.
Therefore, the zero subspace has dimension zero.

Example

Since $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, $\mathbb{R}^n$ has dimension n.
This is why the Cartesian plane, $\mathbb{R}^2$, is called 2-dimensional, and $\mathbb{R}^3$ is called 3-dimensional.

Problem

Let

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}.$$

Show that U is a subspace of $\mathbb{R}^4$, find a basis of U, and find dim(U).

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$U = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; b, c, d \in \mathbb{R} \right\}$$

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$U = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; b, c, d \in \mathbb{R} \right\}$$

This shows that U is a subspace of $\mathbb{R}^4$, since $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where

$$\begin{aligned} \vec{x}_1 &= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T \\ \vec{x}_2 &= \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \\ \vec{x}_3 &= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

Solution (continued)

Furthermore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon (RRE) form of the matrix whose columns are $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution (continued)

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is linearly independent, as can be seen by taking the reduced row-echelon (RRE) form of the matrix whose columns are $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since every column of the RRE matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is linearly independent and spans U, so is a basis of U, and hence U has dimension three. ■

Example (Important!)

Suppose that $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ is a basis of $\mathbb{R}^n$ and that A is an $n \times n$ invertible matrix. Let $D = \{A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n\}$, and let

$$X = \left[\begin{array}{cccc} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{array}\right].$$

Since B is a basis of $\mathbb{R}^n$, B is independent (also a spanning set of $\mathbb{R}^n$); thus X is invertible. Now, because A and X are invertible, so is

$$AX = \left[\begin{array}{cccc} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{array}\right].$$

Therefore, the columns of AX are independent and span $\mathbb{R}^n$. Since the columns of AX are the vectors of D, D is a basis of $\mathbb{R}^n$.

Finding Bases and Dimension

Finding Bases and Dimension

Theorem

Let U be a subspace of $\mathbb{R}^n$. Then

1. U has a basis, and $\dim(U) \leq n$.
2. Any independent set of U can be extended (by adding vectors) to a basis of U.
3. Any spanning set of U can be cut down (by deleting vectors) to a basis of U.

Example

Previously, we showed that

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

is a subspace of $\mathbb{R}^4$, and that $\dim(U) = 3$.

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$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

is a subspace of $\mathbb{R}^4$, and that $\dim(U) = 3$. Also, it is easy to verify that

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of U.

Example

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$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

is a subspace of $\mathbb{R}^4$, and that $\dim(U) = 3$. Also, it is easy to verify that

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of U.

By a previous Theorem, S can be extended to a basis of U. To do so, find a vector in U that is not in span(S).

Example (continued)

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

Example (continued)

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example (continued)

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

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Therefore, S can be extended to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\} \text{ of U.}$$

Problem

Let

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 4 \\ 4 \\ 11 \\ -3 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix},$$

and let $U = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. Find a basis of U that is a subset of $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, and find $\dim(U)$.

Problem

Let

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 4 \\ 4 \\ 11 \\ -3 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix},$$

and let $U = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$. Find a basis of U that is a subset of $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, and find $\dim(U)$.

Solution

Suppose $a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 + a_4\vec{u}_4 = \vec{0}$. Solve for a_1, a_2, a_3, a_4; if some $a_i \neq 0$, $1 \leq i \leq 4$, then $\vec{u}_i$ can be removed from the set $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$, and the resulting set still spans U. Repeat this on the resulting set until a linearly independent set is obtained.

One solution is $B = \{\vec{u}_1, \vec{u}_2\}$. Then $U = \text{span}(B)$ and B is linearly independent. Therefore B is a basis of U, and thus $\dim(U) = 2$. ■

Remark

In the next section, we will learn an efficient technique for solving this type of problem.

Theorem

Let U be a subspace of $\mathbb{R}^n$ with $\dim(U) = m$, and let $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a subset of U. Then B is linearly independent if and only if B spans U.

Theorem

Let U be a subspace of $\mathbb{R}^n$ with $\dim(U) = m$, and let $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a subset of U. Then B is linearly independent if and only if B spans U.

Proof.

($\Rightarrow$) Suppose B is linearly independent. If $\text{span}(B) \neq U$, then extend B to a basis B' of U by adding appropriate vectors from U. Then B' is a basis of size more than $m = \dim(U)$, which is impossible. Therefore, $\text{span}(B) = U$, and hence B is a basis of U.

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Let U be a subspace of $\mathbb{R}^n$ with $\dim(U) = m$, and let $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a subset of U. Then B is linearly independent if and only if B spans U.

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($\Rightarrow$) Suppose B is linearly independent. If $\text{span}(B) \neq U$, then extend B to a basis B' of U by adding appropriate vectors from U. Then B' is a basis of size more than $m = \dim(U)$, which is impossible. Therefore, $\text{span}(B) = U$, and hence B is a basis of U.

($\Leftarrow$) Conversely, suppose $\text{span}(B) = U$. If B is not linearly independent, then cut B down to a basis B' of U by deleting appropriate vectors. But then B' is a basis of size less than $m = \dim(U)$, which is impossible. Therefore, B is linearly independent, and hence B is a basis of U. ■

Remark

Let U be a subspace of $\mathbb{R}^n$ and suppose $B \subseteq U$.

- If B spans U and $|B| = \dim(U)$, then B is also independent, and hence B is a basis of U.
- If B is independent and $|B| = \dim(U)$, then B also spans U, and hence B is a basis of U.

Therefore, if $|B| = \dim(U)$, in order to prove that B is a basis, it is sufficient to prove either of the following two statements:

1. B is independent
2. B spans U

Theorem

Let U and W be subspace of $\mathbb{R}^{n}$, and suppose that $\mathrm{U} \subseteq \mathrm{W}$. Then

1. $\dim(\mathrm{U}) \leq \dim(\mathrm{W})$.
2. If $\dim(\mathrm{U}) = \dim(\mathrm{W})$, then $\mathrm{U} = \mathrm{W}$.

Theorem

Let U and W be subspace of $\mathbb{R}^n$, and suppose that $U \subseteq W$. Then

1. $\dim(U) \leq \dim(W)$.
2. If $\dim(U) = \dim(W)$, then $U = W$.

Proof.

Let $\dim(W) = k$, and let B be a basis of U.

1. If $\dim(U) > k$, then B is a subset of independent vectors of W with $|B| = \dim(U) > k$, which contradicts the Fundamental Theorem.

Theorem

Let U and W be subspace of $\mathbb{R}^n$, and suppose that $U \subseteq W$. Then

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2. If $\dim(U) = \dim(W)$, then $U = W$.

Proof.

Let $\dim(W) = k$, and let B be a basis of U.

1. If $\dim(U) > k$, then B is a subset of independent vectors of W with $|B| = \dim(U) > k$, which contradicts the Fundamental Theorem.
2. If $\dim(U) = \dim(W)$, then B is an independent subset of W containing $k = \dim(W)$ vectors. Therefore, B spans W, so B is a basis of W, and $U = \mathrm{span}(B) = W$.

■

Example

Any subspace U of $\mathbb{R}^2$, other than $\{\vec{0}_2\}$ and $\mathbb{R}^2$ itself, must have dimension one, and thus has a basis consisting of one nonzero vector, say $\vec{u}$. Thus $U = \text{span}\{\vec{u}\}$, and hence is a line through the origin.

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Example

Any subspace U of $\mathbb{R}^3$, other than $\{\vec{0}_3\}$ and $\mathbb{R}^3$ itself, must have dimension one or two. If $\dim(U) = 1$, then, as in the previous example, U is a line through the origin. Otherwise $\dim(U) = 2$, and U has a basis consisting of two linearly independent vectors, say $\vec{u}$ and $\vec{v}$. Thus $U = \text{span}\{\vec{u}, \vec{v}\}$, and hence is a plane through the origin.