

Math 221: LINEAR ALGEBRA

Chapter 5. Vector Space $\mathbb{R}^n$

§5-4. Rank of a Matrix

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

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Row Space and Column Spaces

Definitions

Let A be an $m \times n$ matrix.

- The **column space of A** , denoted $\text{col}(A)$ is the subspace of $\mathbb{R}^m$ spanned by the columns of A .

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

- The **row space of A** , denoted $\text{row}(A)$ is the subspace of $\mathbb{R}^n$ spanned by the rows of A (or the columns of A^T).

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

We saw earlier that $\text{col}(A) = \text{im}(A)$.

Remark (Notation)

Let A and B be $m \times n$ matrices. We write $A \rightarrow B$ if B can be obtained from A by a sequence of elementary row (column) operations. Note that $A \rightarrow B$ if and only if $B \rightarrow A$.

Lemma

Let A and B be $m \times n$ matrices.

1. If $A \rightarrow B$ by elementary row operations, then $\text{row}(A) = \text{row}(B)$.
2. If $A \rightarrow B$ by elementary column operations, then $\text{col}(A) = \text{col}(B)$.

Proof.

It suffices to prove only part one, and only for a single row operation.

(Why?)

Thus let $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$ denote the rows of A .

- If B is obtained from A by interchanging two rows of A , then A and B have exactly the same rows, so $\text{row}(B) = \text{row}(A)$.

Proof. (continued)

- Suppose $p \neq 0$, and suppose that for some j , $1 \leq j \leq m$, B is obtained from A by multiplying row j by p . Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \dots, p\vec{r}_j, \dots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \dots, p\vec{r}_j, \dots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since

$$\{\vec{r}_1, \dots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

Proof. (continued)

- Suppose $p \neq 0$, and suppose that for some i and j , $1 \leq i, j \leq m$, B is obtained from A by adding p times row j to row i . Without loss of generality, we may assume $i < j$. Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \dots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \dots, \vec{r}_j, \dots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \dots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \dots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since

$$\{\vec{r}_1, \dots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.



Corollary

Let A be an $m \times n$ matrix, U an invertible $m \times m$ matrix, and V an invertible $n \times n$ matrix. Then $\text{row}(UA) = \text{row}(A)$ and $\text{col}(AV) = \text{col}(A)$,

Proof.

Since U is invertible, U is a product of elementary matrices, implying that $A \rightarrow UA$ by a sequence of elementary row operations. By Lemma 2, $\text{row}(UA) = \text{row}(A)$.

Now consider AV : $\text{col}(AV) = \text{row}((AV)^T) = \text{row}(V^T A^T)$ and V^T is invertible (a matrix is invertible if and only if its transpose is invertible). It follows from the first part of this Corollary that

$$\text{row}(V^T A^T) = \text{row}(A^T).$$

But $\text{row}(A^T) = \text{col}(A)$, and therefore $\text{col}(AV) = \text{col}(A)$. ■

Lemma

If R is a row-echelon matrix then

1. the nonzero rows of R are a basis of $\text{row}(R)$;
2. the columns of R containing the leading ones are a basis of $\text{col}(R)$.

Example

Let

$$R = \begin{bmatrix} 1 & 2 & 2 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Since the nonzero rows of R are linearly independent, they form a basis of $\text{row}(R)$.
2. Let $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathbb{R}^5$. Then B is linearly independent and spans $\text{col}(R)$, and thus is a basis of $\text{col}(R)$. This tells us that $\dim(\text{col}(R)) = 4$. Now let X denote the set of columns of R that contain the leading ones. Then X is a linearly independent subset of $\text{col}(R)$ with $4 = \dim(\text{col}(R))$ vectors. It follows that X spans $\text{col}(R)$, and therefore is a basis of $\text{col}(R)$.

Problem

Find a basis of $U = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$ and find $\dim(U)$.

Solution

Let A be the 3×4 matrix whose **rows** are the three columns listed. Then $U = \text{row}(A)$, so it suffices to find a basis of $\text{row}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix}.$$

Find R , a row-echelon form of A . Then the **nonzero rows of R** are a basis of $\text{row}(R)$. Since $\text{row}(A) = \text{row}(R)$, the nonzero rows of R are a basis of $\text{row}(A)$.

Solution (continued)

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 5/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}$ is a basis of U and $\dim(U) = 2$.

Solution (Another solution – usually more work.)

Take a linear combination of the three given vectors and set it equal to $\vec{0}_4$. If the vectors are independent, then they form a basis of U . Otherwise, delete vectors to cut the given set of vectors down to a basis.

The Rank Theorem

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$$

Remark

Recall that $\text{rank}(A)$ is defined to be the nonzero rows in the row echelon form of A . From what we just learned, the **rank of A** can be equivalently defined as **$\text{rank}(A) = \dim(\text{row}(A))$** .

Theorem (Rank Theorem)

Let $A = \begin{bmatrix} \vec{A}_1 & \vec{A}_2 & \cdots & \vec{A}_n \end{bmatrix}$ be an $m \times n$ matrix with columns $\{\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n\}$, and suppose that $\text{rank}(A) = r$. Then

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = r.$$

Furthermore, if R is a row-echelon form of A then

1. the r nonzero rows of R are a basis of $\text{row}(A)$;
2. if $S = \{\vec{A}_{j_1}, \vec{A}_{j_2}, \dots, \vec{A}_{j_r}\}$ are the r columns of A corresponding to the columns of R containing leading ones, then S is basis of $\text{col}(A)$.

Problem

For the following matrix A, find rank (A) and bases for row(A) and col(A).

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

► rank (A) = 2.

► $\left\{ \begin{bmatrix} 1 & -2 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -1 & 2 \end{bmatrix} \right\}$ is a basis of row(A).

► $\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -5 \\ -1 \end{bmatrix} \right\}$ is a basis of col(A).



Problem (revisited)

Find a basis of $U = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$ and find $\dim(U)$.

Solution

Let A denote the matrix whose columns are the three vectors listed, and let R denote a row-echelon form of A . Then

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -1 \\ 0 & 5 & 5 \\ 3 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

By the Rank Theorem, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\}$ is a basis of $U = \text{col}(A)$, so

$$\dim(U) = 2.$$



Compare this to the basis found earlier.

Corollary

1. For any matrix A , $\text{rank}(A) = \text{rank}(A^T)$.
2. For any $m \times n$ matrix A , $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
3. Let A be an $m \times n$ matrix. If U and V are invertible matrices (of sizes $m \times m$ and $n \times n$, respectively), then

$$\text{rank}(A) = \text{rank}(UA) = \text{rank}(AV).$$

Lemma

Let A be an $m \times n$ matrix, U a $p \times m$ matrix, and V an $n \times q$ matrix.

1. $\text{col}(AV) \subseteq \text{col}(A)$ with equality if $VV' = I_n$ for some V' .
2. $\text{row}(UA) \subseteq \text{row}(A)$ with equality if $U'U = I_m$ for some U' .

Proof.

(1) Write $V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_q]$, where $\vec{v}_j$ denotes column j of V , $1 \leq j \leq q$. Then $AV = [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_q]$, where $A\vec{v}_j$ is column j of AV . By the definition of matrix-vector multiplication, $A\vec{v}_j$ is a linear combination of the columns of A , and thus $A\vec{v}_j \in \text{col}(A)$ for each j . Since $A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q \in \text{col}(A)$,

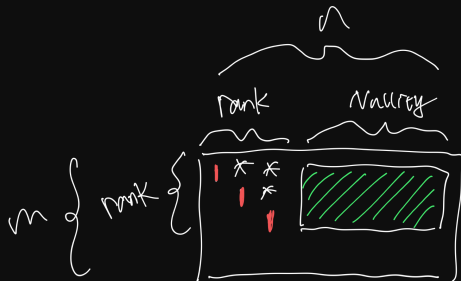
$$\text{span}\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q\} \subseteq \text{col}(A),$$

i.e., $\text{col}(AV) \subseteq \text{col}(A)$. If for some V' we have $VV' = I_n$, then

$$\text{col}(A) = \text{col}(AVV') \subseteq \text{col}(AV) \subseteq \text{col}(A).$$

(2) This can be proved by part (1) and the fact that $\text{row}(A) = \text{col}(A^T)$. ■

Rank-Nullity Theorem



Theorem (Rank-Nullity Theorem)

Let A denote an $m \times n$ matrix of rank r . Then

1. The $n - r$ basic solutions to the system $A\vec{x} = \vec{0}_m$ provided by the Gaussian algorithm are a basis of $\text{null}(A)$, so

$$\dim(\text{null}(A)) = n - r.$$

2. The rank theorem provides a basis of $\text{im}(A) = \text{col}(A)$, and $\dim(\text{im}(A)) = r$.

Remark (Common notation)

The nullspace A is also called kernel space of A , written as $\ker(A)$, i.e., $\ker(A) = \text{null}(A)$. Usually, the **nullity** of A is defined to be

$$\text{Nullity}(A) = \dim(\text{null}(A)) = \dim(\ker(A))$$

Let $T : V \mapsto W$ be the linear map from space V to W . Suppose $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ and let A be the induced matrix.

$$\begin{array}{ccccc}
 \text{Rank}(T) & + & \text{Nullity}(T) & = & \dim(V) \\
 \parallel & & \parallel & & \parallel \\
 \text{Rank}(A) & & \text{Nullity}(A) & & \dim(\mathbb{R}^n) \\
 \parallel & & \parallel & & \parallel \\
 \dim(\text{im}(A)) & & \dim(\text{null}(A)) & & n \\
 \parallel & & \parallel & & \\
 r & & \dim(\ker(A)) & &
 \end{array}$$

Proof. (Outline)

- ▶ We have already seen that $\text{null}(A)$ is spanned by any set of basic solutions to $A\vec{x} = \vec{0}_m$, so it is enough to prove that $\dim(\text{null}(A)) = n - r$, which will imply that the set of basic solutions is independent, hence this set forms a basis.
- ▶ Suppose $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is a basis of $\text{null}(A)$
- ▶ Extend $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ to a basis $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \dots, \vec{x}_n\}$ of $\mathbb{R}^n$.
- ▶ Consider the set $\{A\vec{x}_1, A\vec{x}_2, \dots, A\vec{x}_k, \dots, A\vec{x}_n\} \subseteq \mathbb{R}^m$
- ▶ Then $A\vec{x}_j = \vec{0}_m$ for $1 \leq j \leq k$ since $\vec{x}_1, \dots, \vec{x}_k \in \text{null}(A)$.
- ▶ To complete the proof, show $S = \{A\vec{x}_{k+1}, \dots, A\vec{x}_n\}$ is a basis of $\text{im}(A)$, by showing that (exercise!)
 - (1) S is independent
 - (2) S spans $\text{im}(A)$
- ▶ Since $\text{im}(A) = \text{col}(A)$, $\dim(\text{im}(A)) = r$, implying $n - k = r$. Hence $k = n - r$. 

Problem

For the following matrix A , find bases for $\text{null}(A)$ and $\text{im}(A)$, and find their dimensions.

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

Solution

Find the basic solutions to $A\vec{x} = \vec{0}_4$.

$$\left[\begin{array}{cccc|c} 2 & -4 & 6 & 8 & 0 \\ 2 & -1 & 3 & 2 & 0 \\ 4 & -5 & 9 & 10 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Hence,

$$\vec{x} = \begin{bmatrix} -s \\ s + 2t \\ s \\ t \end{bmatrix} \quad s, t \in \mathbb{R}.$$

Therefore,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -5 \\ -1 \end{bmatrix} \right\}$$

are bases of $\text{null}(A)$ and $\text{im}(A)$, respectively, so

$$\dim(\text{null}(A)) = 2 \quad \text{and} \quad \dim(\text{im}(A)) = 2.$$



Problem

Can a 5×6 matrix have independent columns? Independent rows? Justify your answer.

Solution

The rank of the matrix is at most five; since there are six columns, **the columns can not be independent**. However, the rows could be independent: take a 5×6 matrix whose first five columns are the columns of the 5×5 identity matrix.

Problem

Let A be an $m \times n$ matrix with $\text{rank}(A) = m$. Prove that $m \leq n$.

Proof.

As a consequence of the Rank Theorem, we have

$$\text{rank}(A) \leq m \quad \text{and} \quad \text{rank}(A) \leq n.$$

Since $\text{rank}(A) = m$, it follows that $m \leq n$. ■

Problem

Let A be an 5×9 matrix. Is it possible that $\dim(\text{null}(A)) = 3$? Justify your answer.

Solution

As a consequence of the Rank Theorem, we have $\text{rank}(A) \leq 5$, so $\dim(\text{im}(A)) \leq 5$. Since $\dim(\text{null}(A)) = 9 - \dim(\text{im}(A))$, it follows that

$$\dim(\text{null}(A)) \geq 9 - 5 = 4.$$

Therefore, it is not possible that $\dim(\text{null}(A)) = 3$.



Full Rank Cases

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$.
3. The columns of A are independent in $\mathbb{R}^m$.
4. The $n \times n$ matrix $A^T A$ is invertible.
5. There exists an $n \times m$ matrix C so that $CA = I_n$.
6. If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.

Theorem

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$.
3. The rows of A are independent in $\mathbb{R}^n$.
4. The $m \times m$ matrix AA^T is invertible.
5. There exists an $n \times m$ matrix C so that $AC = I_m$.
6. The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

Problem

Let $\vec{x} = (x_1, \dots, x_k)^T \in \mathbb{R}^k$. Show that the following matrix is invertible if and only if $\{x_i, i = 1, \dots, k\}$ are not all equal:

$$\begin{pmatrix} k & x_1 + \dots + x_k \\ x_1 + \dots + x_k & ||x||^2 \end{pmatrix}$$

Solution

Notice that

$$\begin{pmatrix} k & x_1 + \dots + x_k \\ x_1 + \dots + x_k & ||x||^2 \end{pmatrix} = A^T A$$

with

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_k \end{bmatrix}.$$

Now $A^T A$ is invertible iff the two columns of A are independent iff $\{x_i, i = 1, \dots, k\}$ are not all equal. ■