

# Math 221: LINEAR ALGEBRA

## Chapter 5. Vector Space $\mathbb{R}^n$

### §5-4. Rank of a Matrix

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<sup>1</sup>Slides are adapted from those by Karen Seyffarth from University of Calgary.

Row Space and Column Spaces

The Rank Theorem

Rank-Nullity Theorem

Full Rank Cases



# Row Space and Column Spaces

## Definitions

Let  $A$  be an  $m \times n$  matrix.

- ▶ The **column space of  $A$** , denoted  $\text{col}(A)$  is the subspace of  $\mathbb{R}^m$  spanned by the columns of  $A$ .

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

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- ▶ The **row space of  $A$** , denoted  $\text{row}(A)$  is the subspace of  $\mathbb{R}^n$  spanned by the rows of  $A$  (or the columns of  $A^T$ ).

$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$



We saw earlier that  $\text{col}(A) = \text{im}(A)$ .

### Remark ( Notation )

Let  $A$  and  $B$  be  $m \times n$  matrices. We write  $A \rightarrow B$  if  $B$  can be obtained from  $A$  by a sequence of elementary row (column) operations. Note that  $A \rightarrow B$  if and only if  $B \rightarrow A$ .

## Lemma

Let  $A$  and  $B$  be  $m \times n$  matrices.

1. If  $A \rightarrow B$  by elementary row operations, then  $\text{row}(A) = \text{row}(B)$ .
2. If  $A \rightarrow B$  by elementary column operations, then  $\text{col}(A) = \text{col}(B)$ .



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## Proof.

It suffices to prove only part one, and only for a single row operation.  
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Thus let  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$  denote the rows of  $A$ .

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## Proof.

It suffices to prove only part one, and only for a single row operation.

(Why?)

Thus let  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_m$  denote the rows of  $A$ .

- If  $B$  is obtained from  $A$  by interchanging two rows of  $A$ , then  $A$  and  $B$  have exactly the same rows, so  $\text{row}(B) = \text{row}(A)$ .

Proof. (continued)

- ▶ Suppose  $p \neq 0$ , and suppose that for some  $j$ ,  $1 \leq j \leq m$ ,  $B$  is obtained from  $A$  by multiplying row  $j$  by  $p$ . Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \dots, p\vec{r}_j, \dots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \dots, p\vec{r}_j, \dots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that  $\text{row}(B) \subseteq \text{row}(A)$ .

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Since

$$\{\vec{r}_1, \dots, p\vec{r}_j, \dots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that  $\text{row}(B) \subseteq \text{row}(A)$ . Conversely, since

$$\{\vec{r}_1, \dots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that  $\text{row}(A) \subseteq \text{row}(B)$ . Therefore,  $\text{row}(B) = \text{row}(A)$ .

Proof. (continued)

- ▶ Suppose  $p \neq 0$ , and suppose that for some  $i$  and  $j$ ,  $1 \leq i, j \leq m$ ,  $B$  is obtained from  $A$  by adding  $p$  times row  $j$  to row  $i$ . Without loss of generality, we may assume  $i < j$ .

Proof. (continued)

- ▶ Suppose  $p \neq 0$ , and suppose that for some  $i$  and  $j$ ,  $1 \leq i, j \leq m$ ,  $B$  is obtained from  $A$  by adding  $p$  times row  $j$  to row  $i$ . Without loss of generality, we may assume  $i < j$ . Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \dots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \dots, \vec{r}_j, \dots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \dots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \dots, \vec{r}_m\} \subseteq \text{row}(A),$$

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$$\text{row}(B) = \text{span}\{\vec{r}_1, \dots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \dots, \vec{r}_j, \dots, \vec{r}_m\}.$$

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it follows that  $\text{row}(A) \subseteq \text{row}(B)$ . Therefore,  $\text{row}(B) = \text{row}(A)$ .





## Corollary

Let  $A$  be an  $m \times n$  matrix,  $U$  an invertible  $m \times m$  matrix, and  $V$  an invertible  $n \times n$  matrix. Then  $\text{row}(UA) = \text{row}(A)$  and  $\text{col}(AV) = \text{col}(A)$ ,

## Corollary

Let  $A$  be an  $m \times n$  matrix,  $U$  an invertible  $m \times m$  matrix, and  $V$  an invertible  $n \times n$  matrix. Then  $\text{row}(UA) = \text{row}(A)$  and  $\text{col}(AV) = \text{col}(A)$ ,

## Proof.

Since  $U$  is invertible,  $U$  is a product of elementary matrices, implying that  $A \rightarrow UA$  by a sequence of elementary row operations. By Lemma 2,  $\text{row}(UA) = \text{row}(A)$ .

Now consider  $AV$ :  $\text{col}(AV) = \text{row}((AV)^T) = \text{row}(V^T A^T)$  and  $V^T$  is invertible (a matrix is invertible if and only if its transpose is invertible). It follows from the first part of this Corollary that

$$\text{row}(V^T A^T) = \text{row}(A^T).$$

But  $\text{row}(A^T) = \text{col}(A)$ , and therefore  $\text{col}(AV) = \text{col}(A)$ . ■

## Lemma

If  $\mathbf{R}$  is a row-echelon matrix then

1. the nonzero rows of  $\mathbf{R}$  are a basis of  $\text{row}(\mathbf{R})$ ;
2. the columns of  $\mathbf{R}$  containing the leading ones are a basis of  $\text{col}(\mathbf{R})$ .

## Example

Let

$$R = \begin{bmatrix} 1 & 2 & 2 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Since the nonzero rows of  $R$  are linearly independent, they form a basis of  $\text{row}(R)$ .

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1. Since the nonzero rows of  $R$  are linearly independent, they form a basis of  $\text{row}(R)$ .
2. Let  $B = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\} \subseteq \mathbb{R}^5$ . Then  $B$  is linearly independent and spans  $\text{col}(R)$ , and thus is a basis of  $\text{col}(R)$ . **This tells us that  $\dim(\text{col}(R)) = 4$ .** Now let  $X$  denote the set of columns of  $R$  that contain the leading ones.

## Example

Let

$$R = \begin{bmatrix} 1 & 2 & 2 & -2 & 0 & 0 \\ 0 & 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. Since the nonzero rows of  $R$  are linearly independent, they form a basis of  $\text{row}(R)$ .
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### Problem

Find a basis of  $U = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$  and find  $\dim(U)$ .

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## Solution

Let  $A$  be the  $3 \times 4$  matrix whose **rows** are the three columns listed. Then  $U = \text{row}(A)$ , so it suffices to find a basis of  $\text{row}(A)$ .

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix}.$$

Find  $R$ , a row-echelon form of  $A$ . Then the **nonzero rows of  $R$**  are a basis of  $\text{row}(R)$ . Since  $\text{row}(A) = \text{row}(R)$ , the nonzero rows of  $R$  are a basis of  $\text{row}(A)$ .



Solution (continued)

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 5/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,  $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}$  is a basis of  $U$  and  $\dim(U) = 2$ .

Solution (continued)

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 5 & 1 \\ 4 & -1 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 5/3 & -5/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore,  $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 5 \\ -5 \end{bmatrix} \right\}$  is a basis of  $U$  and  $\dim(U) = 2$ .

Solution (Another solution – usually more work.)

Take a linear combination of the three given vectors and set it equal to  $\vec{0}_4$ . If the vectors are independent, then they form a basis of  $U$ . Otherwise, delete vectors to cut the given set of vectors down to a basis.



# The Rank Theorem

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$$

## Remark

Recall that  $\text{rank}(A)$  is defined to be the nonzero rows in the row echelon form of  $A$ . From what we just learned, the **rank of  $A$**  can be equivalently defined as  **$\text{rank}(A) = \dim(\text{row}(A))$** .

## Remark

Recall that  $\text{rank}(A)$  is defined to be the nonzero rows in the row echelon form of  $A$ . From what we just learned, the **rank of  $A$**  can be equivalently defined as  **$\text{rank}(A) = \dim(\text{row}(A))$** .

## Theorem (Rank Theorem)

Let  $A = \begin{bmatrix} \vec{A}_1 & \vec{A}_2 & \cdots & \vec{A}_n \end{bmatrix}$  be an  $m \times n$  matrix with columns  $\{\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n\}$ , and suppose that  $\text{rank}(A) = r$ . Then

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = r.$$

Furthermore, if  $R$  is a row-echelon form of  $A$  then

1. the  $r$  nonzero rows of  $R$  are a basis of  $\text{row}(A)$ ;
2. if  $S = \{\vec{A}_{j_1}, \vec{A}_{j_2}, \dots, \vec{A}_{j_r}\}$  are the  $r$  columns of  $A$  corresponding to the columns of  $R$  containing leading ones, then  $S$  is basis of  $\text{col}(A)$ .

## Problem

For the following matrix  $A$ , find  $\text{rank}(A)$  and bases for  $\text{row}(A)$  and  $\text{col}(A)$ .

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

## Problem

For the following matrix A, find rank (A) and bases for row(A) and col(A).

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

## Solution

$$\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



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►  $\text{rank}(A) = 2$ .

## Problem

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►  $\text{rank}(A) = 2$ .

►  $\{[1 \ -2 \ 3 \ 4], [0 \ -1 \ -1 \ 2]\}$  is a basis of  $\text{row}(A)$ .

## Problem

For the following matrix A, find rank (A) and bases for row(A) and col(A).

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

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$$\begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

► rank (A) = 2.

►  $\{ [ 1 \ -2 \ 3 \ 4 ], [ 0 \ -1 \ -1 \ 2 ] \}$  is a basis of row(A).

►  $\left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -5 \\ -1 \end{bmatrix} \right\}$  is a basis of col(A). ■

Problem (revisited)

Find a basis of  $U = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$  and find  $\dim(U)$ .

## Problem (revisited)

Find a basis of  $U = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \\ 7 \end{bmatrix} \right\}$  and find  $\dim(U)$ .

## Solution

Let  $A$  denote the matrix whose columns are the three vectors listed, and let  $R$  denote a row-echelon form of  $A$ . Then

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & -1 \\ 0 & 5 & 5 \\ 3 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.$$

By the Rank Theorem,  $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\}$  is a basis of  $U = \text{col}(A)$ , so

$$\dim(U) = 2. \quad \blacksquare$$

Compare this to the basis found earlier.

## Corollary

1. For any matrix  $A$ ,  $\text{rank}(A) = \text{rank}(A^T)$ .
2. For any  $m \times n$  matrix  $A$ ,  $\text{rank}(A) \leq m$  and  $\text{rank}(A) \leq n$ .
3. Let  $A$  be an  $m \times n$  matrix. If  $U$  and  $V$  are invertible matrices (of sizes  $m \times m$  and  $n \times n$ , respectively), then

$$\text{rank}(A) = \text{rank}(UA) = \text{rank}(AV).$$

## Lemma

Let  $A$  be an  $m \times n$  matrix,  $U$  a  $p \times m$  matrix, and  $V$  an  $n \times q$  matrix.

1.  $\text{col}(AV) \subseteq \text{col}(A)$  with equality if  $VV' = I_n$  for some  $V'$ .
2.  $\text{row}(UA) \subseteq \text{row}(A)$  with equality if  $U'U = I_m$  for some  $U'$ .

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2.  $\text{row}(UA) \subseteq \text{row}(A)$  with equality if  $U'U = I_m$  for some  $U'$ .

## Proof.

(1) Write  $V = [ \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_q ]$ , where  $\vec{v}_j$  denotes column  $j$  of  $V$ ,  $1 \leq j \leq q$ . Then  $AV = [ A\vec{v}_1 \quad A\vec{v}_2 \quad \cdots \quad A\vec{v}_q ]$ , where  $A\vec{v}_j$  is column  $j$  of  $AV$ . By the definition of matrix-vector multiplication,  $A\vec{v}_j$  is a linear combination of the columns of  $A$ , and thus  $A\vec{v}_j \in \text{col}(A)$  for each  $j$ . Since  $A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q \in \text{col}(A)$ ,

$$\text{span}\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q\} \subseteq \text{col}(A),$$

i.e.,  $\text{col}(AV) \subseteq \text{col}(A)$ .



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## Proof.

(1) Write  $V = [ \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_q ]$ , where  $\vec{v}_j$  denotes column  $j$  of  $V$ ,  $1 \leq j \leq q$ . Then  $AV = [ A\vec{v}_1 \quad A\vec{v}_2 \quad \cdots \quad A\vec{v}_q ]$ , where  $A\vec{v}_j$  is column  $j$  of  $AV$ . By the definition of matrix-vector multiplication,  $A\vec{v}_j$  is a linear combination of the columns of  $A$ , and thus  $A\vec{v}_j \in \text{col}(A)$  for each  $j$ . Since  $A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q \in \text{col}(A)$ ,

$$\text{span}\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q\} \subseteq \text{col}(A),$$

i.e.,  $\text{col}(AV) \subseteq \text{col}(A)$ . If for some  $V'$  we have  $VV' = I_n$ , then

$$\text{col}(A) = \text{col}(AVV') \subseteq \text{col}(AV) \subseteq \text{col}(A).$$

## Lemma

Let  $A$  be an  $m \times n$  matrix,  $U$  a  $p \times m$  matrix, and  $V$  an  $n \times q$  matrix.

1.  $\text{col}(AV) \subseteq \text{col}(A)$  with equality if  $VV' = I_n$  for some  $V'$ .
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$$\text{span}\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_q\} \subseteq \text{col}(A),$$

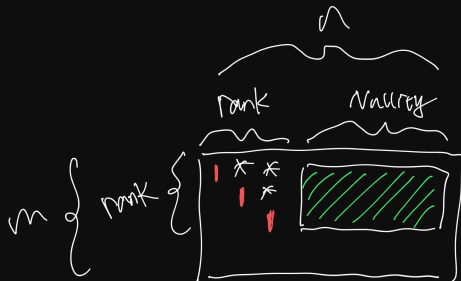
i.e.,  $\text{col}(AV) \subseteq \text{col}(A)$ . If for some  $V'$  we have  $VV' = I_n$ , then

$$\text{col}(A) = \text{col}(AVV') \subseteq \text{col}(AV) \subseteq \text{col}(A).$$

(2) This can be proved by part (1) and the fact that  $\text{row}(A) = \text{col}(A^T)$ . ■



# Rank-Nullity Theorem



## Theorem (Rank-Nullity Theorem)

Let  $A$  denote an  $m \times n$  matrix of rank  $r$ . Then

1. The  $n - r$  basic solutions to the system  $A\vec{x} = \vec{0}_m$  provided by the Gaussian algorithm are a basis of  $\text{null}(A)$ , so

$$\dim(\text{null}(A)) = n - r.$$

2. The rank theorem provides a basis of  $\text{im}(A) = \text{col}(A)$ , and  $\dim(\text{im}(A)) = r$ .

## Theorem (Rank-Nullity Theorem)

Let  $A$  denote an  $m \times n$  matrix of rank  $r$ . Then

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## Remark (Common notation)

The nullspace  $A$  is also called kernel space of  $A$ , written as  $\ker(A)$ , i.e.,  $\ker(A) = \text{null}(A)$ . Usually, the **nullity** of  $A$  is defined to be

$$\text{Nullity}(A) = \dim(\text{null}(A)) = \dim(\ker(A))$$

Let  $T : V \mapsto W$  be the linear map from space  $V$  to  $W$ . Suppose  $V = \mathbb{R}^n$  and  $W = \mathbb{R}^m$  and let  $A$  be the induced matrix.

$$\begin{array}{rcccl}
 \text{Rank}(T) & + & \text{Nullity}(T) & = & \text{dim}(V) \\
 \parallel & & \parallel & & \parallel \\
 \text{Rank}(A) & & \text{Nullity}(A) & & \text{dim}(\mathbb{R}^n) \\
 \parallel & & \parallel & & \parallel \\
 \text{dim}(\text{im}(A)) & & \text{dim}(\text{null}(A)) & & n \\
 \parallel & & \parallel & & \\
 r & & \text{dim}(\text{ker}(A)) & & 
 \end{array}$$





## Proof. (Outline)

- ▶ We have already seen that  $\text{null}(A)$  is spanned by any set of basic solutions to  $A\vec{x} = \vec{0}_m$ , so it is enough to prove that  $\dim(\text{null}(A)) = n - r$ , which will imply that the set of basic solutions is independent, hence this set forms a basis.
- ▶ Suppose  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  is a basis of  $\text{null}(A)$
- ▶ Extend  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$  to a basis  $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k, \dots, \vec{x}_n\}$  of  $\mathbb{R}^n$ .
- ▶ Consider the set  $\{A\vec{x}_1, A\vec{x}_2, \dots, A\vec{x}_k, \dots, A\vec{x}_n\} \subseteq \mathbb{R}^m$
- ▶ Then  $A\vec{x}_j = \vec{0}_m$  for  $1 \leq j \leq k$  since  $\vec{x}_1, \dots, \vec{x}_k \in \text{null}(A)$ .
- ▶ To complete the proof, show  $S = \{A\vec{x}_{k+1}, \dots, A\vec{x}_n\}$  is a basis of  $\text{im}(A)$ , by showing that (exercise!)
  - (1)  $S$  is independent
  - (2)  $S$  spans  $\text{im}(A)$
- ▶ Since  $\text{im}(A) = \text{col}(A)$ ,  $\dim(\text{im}(A)) = r$ , implying  $n - k = r$ . Hence  $k = n - r$ . ■

## Problem

For the following matrix  $A$ , find bases for  $\text{null}(A)$  and  $\text{im}(A)$ , and find their dimensions.

$$A = \begin{bmatrix} 2 & -4 & 6 & 8 \\ 2 & -1 & 3 & 2 \\ 4 & -5 & 9 & 10 \\ 0 & -1 & 1 & 2 \end{bmatrix}.$$

## Solution

Find the basic solutions to  $A\vec{x} = \vec{0}_4$ .

$$\left[ \begin{array}{cccc|c} 2 & -4 & 6 & 8 & 0 \\ 2 & -1 & 3 & 2 & 0 \\ 4 & -5 & 9 & 10 & 0 \\ 0 & -1 & 1 & 2 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & -2 & 3 & 4 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Hence,

$$\vec{x} = \begin{bmatrix} -s \\ s + 2t \\ s \\ t \end{bmatrix} \quad s, t \in \mathbb{R}.$$

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Therefore,

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -5 \\ -1 \end{bmatrix} \right\}$$

are bases of  $\text{null}(A)$  and  $\text{im}(A)$ , respectively, so

$$\dim(\text{null}(A)) = 2 \quad \text{and} \quad \dim(\text{im}(A)) = 2.$$



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Can a  $5 \times 6$  matrix have independent columns? Independent rows? Justify your answer.

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The rank of the matrix is at most five; since there are six columns, **the columns can not be independent**. However, the rows could be independent:

## Problem

Can a  $5 \times 6$  matrix have independent columns? Independent rows? Justify your answer.

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## Problem

Let  $A$  be an  $m \times n$  matrix with  $\text{rank}(A) = m$ . Prove that  $m \leq n$ .

## Proof.

As a consequence of the Rank Theorem, we have

$$\text{rank}(A) \leq m \quad \text{and} \quad \text{rank}(A) \leq n.$$

Since  $\text{rank}(A) = m$ , it follows that  $m \leq n$ . ■

## Problem

Let  $A$  be an  $5 \times 9$  matrix. Is it possible that  $\dim(\text{null}(A)) = 3$ ? Justify your answer.

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Let  $A$  be an  $5 \times 9$  matrix. Is it possible that  $\dim(\text{null}(A)) = 3$ ? Justify your answer.

## Solution

As a consequence of the Rank Theorem, we have  $\text{rank}(A) \leq 5$ , so  $\dim(\text{im}(A)) \leq 5$ . Since  $\dim(\text{null}(A)) = 9 - \dim(\text{im}(A))$ , it follows that

$$\dim(\text{null}(A)) \geq 9 - 5 = 4.$$

Therefore, it is not possible that  $\dim(\text{null}(A)) = 3$ . ■



## Full Rank Cases

### Theorem

Let  $A$  be an  $m \times n$  matrix. The following are equivalent.

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5. There exists an  $n \times m$  matrix  $C$  so that  $CA = I_n$ .
6. If  $A\vec{x} = \vec{0}_m$  for some  $\vec{x} \in \mathbb{R}^n$ , then  $\vec{x} = \vec{0}_n$ .





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3. The rows of  $A$  are independent in  $\mathbb{R}^n$ .
4. The  $m \times m$  matrix  $AA^T$  is invertible.
5. There exists an  $n \times m$  matrix  $C$  so that  $AC = I_m$ .
6. The system  $A\vec{x} = \vec{b}$  is consistent for every  $\vec{b} \in \mathbb{R}^m$ .

## Problem

Let  $\vec{x} = (x_1, \dots, x_k)^T \in \mathbb{R}^k$ . Show that the following matrix is invertible if and only if  $\{x_i, i = 1, \dots, k\}$  are not all equal:

$$\begin{pmatrix} k & x_1 + \dots + x_k \\ x_1 + \dots + x_k & \|\mathbf{x}\|^2 \end{pmatrix}$$

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## Solution

Notice that

$$\begin{pmatrix} k & x_1 + \dots + x_k \\ x_1 + \dots + x_k & \|\vec{x}\|^2 \end{pmatrix} = A^T A$$

with

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_k \end{bmatrix}.$$

Now  $A^T A$  is invertible iff the two columns of  $A$  are independent iff  $\{x_i, i = 1, \dots, k\}$  are not all equal. ■