Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-1. Examples and Basic Properties

Le Chen[1]

Emory University, 2020 Fall

(last updated on 11/03/2020)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

What is a vector space?

Example one – Matrices

Example Two – Polynomials

More Examples

What is a vector space?

1. $\mathbb{R}^n$
2. Polynomials of order at most n:

$$\{a_0 + a_1 x + \cdots + a_n x^n | a_i \in \mathbb{R},\ i = 1, \cdots, n\}$$

3. The set of $m \times n$ matrices.
4. The set of continuous functions on $[0, 1]$, i.e., $C([0, 1])$.
5. The set of functions on $[0, 1]$ having nth continuous derivatives, i.e., $C^n([0, 1])$.

$\vdots \qquad \vdots$

Definition (Vector Space)

Let V be a nonempty set of objects with two operations: vector addition and scalar multiplication. Then V is called a vector space if it satisfies the following Axioms of Addition and the Axioms of Scalar Multiplication. The elements of V are called vectors.

Definition (continued – Axioms of ADDITION)

A1. V is closed under addition.

$\mathbf{v}, \mathbf{w} \in \mathrm{V} \quad \Longrightarrow \quad \mathbf{u} + \mathbf{v} \in \mathrm{V}$

A2. Addition is commutative.

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$.

A3. Addition is associative.

$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$.

A4. Existence of an additive identity.

There exists an element $\mathbf{0}$ in V so that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathrm{V}$.

A5. Existence of an additive inverse.

For each $\mathbf{u} \in \mathrm{V}$ there exists an element $-\mathbf{u} \in \mathrm{V}$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Definition (continued – Axioms of SCALAR MULTIPLICATION)

S1. V is closed under scalar multiplication.
$\mathbf{v} \in \mathrm{V}$ and $\mathrm{k} \in \mathbb{R}$, $\implies \mathrm{k}\mathbf{v} \in \mathrm{V}$.

S2. Scalar multiplication distributes over vector addition.
$\mathrm{a}(\mathbf{u} + \mathbf{v}) = \mathrm{a}\mathbf{u} + \mathrm{a}\mathbf{v}$ for all $\mathrm{a} \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathrm{V}$.

S3. Scalar multiplication distributes over scalar addition.
$(\mathrm{a} + \mathrm{b})\mathbf{u} = \mathrm{a}\mathbf{u} + \mathrm{b}\mathbf{u}$ for all $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ and $\mathbf{u} \in \mathrm{V}$.

S4. Scalar multiplication is associative.
$\mathrm{a}(\mathrm{b}\mathbf{u}) = (\mathrm{ab})\mathbf{u}$ for all $\mathrm{a}, \mathrm{b} \in \mathbb{R}$ and $\mathbf{u} \in \mathrm{V}$.

S5. Existence of a multiplicative identity for scalar multiplication.
$1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathrm{V}$.

Definition (Vector Difference)

Let V be a vector space and $\mathbf{u}, \mathbf{v} \in V$. The difference of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

(where $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$).

Theorem

Let V be a vector space, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $a \in \mathbb{R}$.

1. If $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.
2. The equation $\mathbf{x} + \mathbf{v} = \mathbf{u}$, has a unique solution $\mathbf{x} \in V$ given by $\mathbf{x} = \mathbf{u} - \mathbf{v}$.
3. $a\mathbf{v} = \mathbf{0}$ if and only if $a = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.
5. $(-a)\mathbf{v} = -(a\mathbf{v}) = a(-\mathbf{v})$.

Example One – Matrices

Example

$\mathbb{R}^n$ with matrix addition and scalar multiplication is a vector space.

Example

$\mathbf{M}_{mn}$, the set of all $m \times n$ matrices (of real numbers) with matrix addition and scalar multiplication is a vector space. It is left as an exercise to verify the ten vector space axioms.

Remark

1. Notation: the $m \times n$ matrix of all zeros is written $\mathbf{0}$ or, when the size of the matrix needs to be emphasized, $\mathbf{0}_{mn}$.
2. The vector space $\mathbf{M}_{mn}$ "is the same as" the vector space $\mathbb{R}^{mn}$. We will make this notion more precise later on. For now, notice that an $m \times n$ matrix has mn entries arranged in m rows and n columns, while a vector in $\mathbb{R}^{mn}$ has mn entries arranged in mn rows and 1 column.

Problem

Let V be the set of all 2×2 matrices of real numbers whose entries sum to zero. We use the usual addition and scalar multiplication of $\mathbf{M}_{22}$. Show that V is a vector space.

Solution

The matrices in V may be described as follows:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22} \;\middle|\; a + b + c + d = 0 \right\}.$$

Since we are using the matrix addition and scalar multiplication of $\mathbf{M}_{22}$, it is automatic that addition is commutative and associative, and that scalar multiplication satisfies the two distributive properties, the associative property, and has 1 as an identity element.

What needs to be shown is closure under addition (for all $\mathbf{v}, \mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} \in V$), and closure under scalar multiplication (for all $\mathbf{v} \in V$ and $k \in \mathbb{R}$, $k\mathbf{v} \in V$), as well as showing the existence of an additive identity and additive inverses in the set V.

Solution (continued)

- **Closure under addition:** Suppose

$$A = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} w_2 & x_2 \\ y_2 & x_2 \end{bmatrix}$$

are in V. Then $w_1 + x_1 + y_1 + z_1 = 0, w_2 + x_2 + y_2 + z_2 = 0$, and

$$A + B = \begin{bmatrix} w_1 & x_1 \\ y_1 & z_1 \end{bmatrix} + \begin{bmatrix} w_2 & x_2 \\ y_2 & z_2 \end{bmatrix} = \begin{bmatrix} w_1 + w_2 & x_1 + x_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}.$$

Since

$$\begin{aligned} &(w_1 + w_2) + (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \\ &= (w_1 + x_1 + y_1 + z_1) + (w_2 + x_2 + y_2 + z_2) \\ &= 0 + 0 = 0, \end{aligned}$$

$A + B$ is in V, so V is closed under addition.

Solution (continued)

- **Closure under scalar multiplication:** Suppose $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ is in V and $k \in \mathbb{R}$. Then $w + x + y + z = 0$, and

$$kA = k \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} kw & kx \\ ky & kz \end{bmatrix}.$$

Since

$$kw + kx + ky + kz = k(w + x + y + z) = k(0) = 0,$$

kA is in V, so V is closed under scalar multiplication.

Solution (continued)

- **Existence of an additive identity:** The additive identity of $\mathbf{M}_{22}$ is the 2×2 matrix of zeros,

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

  Since $0 + 0 + 0 + 0 = 0$, $\mathbf{0}$ is in V, and has the required property (as it does in $\mathbf{M}_{22}$).

Solution (continued)

- **Existence of an additive inverse:** Let $A = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be in V.

  Then $w + x + y + z = 0$, and its additive inverse in $\mathbf{M}_{22}$ is

  $$-A = \begin{bmatrix} -w & -x \\ -y & -z \end{bmatrix}.$$

  Since

  $$(-w) + (-x) + (-y) + (-z) = -(w + x + y + x) = -0 = 0,$$

  $-A$ is in V and has the required property (as it does in $\mathbf{M}_{22}$). ■

Problem

Let

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \quad \text{and} \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0. \right\}.$$

We use the usual addition and scalar multiplication of $\mathbf{M}_{22}$. Show that V is NOT a vector space.

Solution

We need to find a counter example that violates some axioms. Indeed, if

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

then $\det(A) = 0$ and $\det(B) = 0$, so $A, B \in V$. However,

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

and $\det(A + B) = -1$, so $A + B \notin V$, i.e., V is not closed under addition. ■

Example Two – Polynomials

Definition

Let $\mathcal{P}$ be the set of all polynomials in x, with real coefficients, and let $p \in \mathcal{P}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

for some integer n.

- The degree of p is the highest power of x with a nonzero coefficient. Note that $p(x) = 0$ has undefined degree.

Definition (continued)

- Addition. Suppose $p, q \in \mathcal{P}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i \quad \text{and} \quad q(x) = \sum_{i=0}^{m} b_i x^i.$$

  We may assume, without loss of generality, that $n \geq m$; for $j = m+1, m+2, \ldots, n-1, n$, we define $b_j = 0$. Then

$$(p+q)(x) = p(x) + q(x) = \sum_{i=0}^{n} (a_i x^i + b_i x^i) = \sum_{i=0}^{n} (a_i + b_i) x^i.$$

Remark

Note that this definition ensures that $\mathcal{P}$ is closed under addition.

Definition (continued)

- Scalar Multiplication. Suppose $p \in \mathcal{P}$ and $k \in \mathbb{R}$. Then

$$p(x) = \sum_{i=0}^{n} a_i x^i,$$

and

$$(kp)(x) = k(p(x)) = \sum_{i=0}^{n} k(a_i x^i) = \sum_{i=0}^{n} (ka_i)x^i.$$

- The zero polynomial is denoted $\mathbf{0}$. Note that $\mathbf{0} = 0$, but we use $\mathbf{0}$ to emphasize that it is the zero vector of $\mathcal{P}$.

Remark

Note that this definition ensures that $\mathcal{P}$ is closed under scalar multiplication.

Example

The set of polynomials $\mathcal{P}$, with addition and scalar multiplication as defined, is a vector space. It is left as an exercise to verify the ten vector space axioms.

Example

For $n \geq 1$, let $\mathcal{P}_n$ denote the set of all polynomials of degree at most n, along with the zero polynomial, with addition and scalar multiplication as in $\mathcal{P}$, i.e.,

$$\mathcal{P}_n = \left\{a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n \mid a_0, a_1, a_2, \ldots, a_{n-1}, a_n \in \mathbb{R}\right\}.$$

Then $\mathcal{P}_n$ is a vector space, and it is left as an exercise to verify the $\mathcal{P}_n$ is closed under addition and scalar multiplication, and satisfies the ten vector space axioms.

More Examples

Problem

Let $\mathrm{V} = \{(\mathrm{x}, \mathrm{y}) \mid \mathrm{x}, \mathrm{y} \in \mathbb{R}\}$, with addition $\oplus$ and scalar multiplication $\odot$ defined as follows:

For $(\mathrm{x}_1, \mathrm{y}_1), (\mathrm{x}_2, \mathrm{y}_2) \in \mathrm{V}$, and $\mathrm{a}, \mathrm{b} \in \mathbb{R}$:

1. Addition. $(\mathrm{x}_1, \mathrm{y}_1) \oplus (\mathrm{x}_2, \mathrm{y}_2) = (\mathrm{x}_1 + \mathrm{x}_2, \mathrm{y}_1 + \mathrm{y}_2 + 1)$.
2. Scalar Multiplication. $\mathrm{a} \odot (\mathrm{x}_1, \mathrm{y}_1) = (\mathrm{a}\mathrm{x}_1, \mathrm{a}\mathrm{y}_1 + \mathrm{a} - 1)$.

Show that V, with addition and scalar multiplication as defined, is a vector space.

Proof.

1. It is clear that V is closed under $\oplus$ and $\odot$, since both operations produce ordered pairs of real numbers.
2. It is routine to verify that $\oplus$ is commutative and associative.
3. What is the additive identity?
4. What is the additive inverse of $(x, y) \in V$?
5. Verify that $(a + b) \odot (x_1, y_1) = (a \odot (x_1, y_1)) \oplus (b \odot (x_1, y_1))$.
6. Verify that $a \odot ((x_1, y_1) \oplus (x_2, y_2)) = (a \odot (x_1, y_1)) \oplus (a \odot (x_2, y_2))$.
7. Verify that $a \odot (b \odot (x_1, y_1)) = (ab) \odot (x_1, y_1)$.
8. Verify that $1 \odot (x, y) = (x, y)$. ■

Problem

Let $\mathbb{R}_+$ be the set of positive reals. Let the addition $\oplus$ and the scalar multiplication $\odot$ defined as follows:

For $x, y \in \mathbb{R}_+$, and $a \in \mathbb{R}$:

1. Addition. $x \oplus y = xy$.
2. Scalar Multiplication. $a \odot x = x^a$.

Prove that $\mathbb{R}_+$ equipped with $\oplus$ and $\odot$ is a vector space.

Proof.

Verify ten properties in the Axioms! ■

Problem

1. Let $C([0, 1])$ be the set of continuous functions defined on $[0, 1]$ equipped with usual addition and scalar multiplication. Prove that $C([0, 1])$ is a vector space.
2. Let $C^n([0, 1])$ be the set of functions that have continuous nth derivatives ($n \geq 0$) defined on $[0, 1]$, equipped with usual addition and scalar multiplication. Prove that $C^n([0, 1])$ is a vector space.

Proof.

Verify ten properties in the Axioms! $\blacksquare$