Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-3. Linear Independence

Le Chen[1]

Emory University, 2020 Fall

(last updated on 11/03/2020)

[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Linear Independence

Definition

Let V be a vector space and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a subset of V. The set S is linearly independent or simply independent if the following condition holds:

$$\boxed{s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k = \mathbf{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_k = 0}$$

i.e., the only linear combination that vanishes is the trivial one. If S is not linearly independent, then S is said to be dependent.

Example

The set $S = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \right\}$ is a dependent subset of $\mathbb{R}^3$ because

$$a \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has nontrivial solutions, for example $a = 2$, $b = 3$ and $c = -1$.

Problem

Is the set $T = \{3x^2 - x + 2, x^2 + x - 1, x^2 - 3x + 4\}$ an independent subset of $\mathcal{P}_2$?

Solution

Suppose $a(3x^2 - x + 2) + b(x^2 + x - 1) + c(x^2 - 3x + 4) = 0$, for some $a, b, c \in \mathbb{R}$. Then

$$x^2(3a + b + c) + x(-a + b - 3c) + (2a - b + 4c) = 0,$$

implying that

$$\begin{aligned} 3a + b + c &= 0 \\ -a + b - 3c &= 0 \\ 2a - b + 4c &= 0 \end{aligned}$$

Solving this linear system of three equations in three variables

$$\left[\begin{array}{rrr|r} 3 & 1 & 1 & 0 \\ -1 & 1 & -3 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is nontrivial solution, T is a dependent subset of $\mathcal{P}_2$. ∎

Problem

Is $U = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$ an independent subset of $\mathbf{M}_{22}$?

Solution

Suppose $a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$.

$$\Downarrow$$

$$\begin{array}{rclcrcl} a+c & = & 0, & & a+b & = & 0, \\ b+c & = & 0, & & a+c & = & 0. \end{array}$$

This system of four equations in three variables has unique solution $a = b = c = 0$,

$$\Downarrow$$

U is an independent subset of $\mathbf{M}_{22}$.

∎

Example (An independent subset of $\mathcal{P}_n$)

Consider $\{1, x, x^2, \ldots, x^n\}$, and suppose that

$$a_0 \cdot 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$$

for some $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then $a_0 = a_1 = \cdots = a_n = 0$, and thus $\{1, x, x^2, \ldots, x^n\}$ is an independent subset of $\mathcal{P}_n$.

Example (Polynomials with distinct degrees)

Any set of polynomials with DISTINCT degrees is independent.

For example,

$$\left\{2x^4 - x^3 + 5, \quad -3x^3 + 2x^2 + 2, \quad 4x^2 + x - 3, \quad 2x - 1, \quad 3\right\}$$

is an independent subset of $\mathcal{P}_4$.

Example

As we saw earlier, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ (the standard basis of $\mathbb{R}^n$) is an independent subset of $\mathbb{R}^n$.

Example

$$U = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is an independent subset of $\mathbf{M}_{32}$.

Example (An independent subset of $\mathbf{M}_{mn}$)

In general, the set of mn $m \times n$ matrices that have a '1' in position (i, j) and zeros elsewhere, $1 \leq i \leq m$, $1 \leq j \leq n$, constitutes an independent subset of $\mathbf{M}_{mn}$.

Example

Let V be a vector space.

1. If $\mathbf{v}$ is a nonzero vector of V, then $\{\mathbf{v}\}$ is an independent subset of V.

   Proof. Suppose that $k\mathbf{v} = \mathbf{0}$ for some $k \in \mathbb{R}$. Since $\mathbf{v} \neq \mathbf{0}$, it must be that $k = 0$, and therefore $\{\mathbf{v}\}$ is an independent set. ∎

2. The zero vector of V, $\mathbf{0}$ is never an element of an independent subset of V.

   Proof. Suppose $S = \{\mathbf{0}, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\}$ is a subset of V. Then

   $$1(\mathbf{0}) + 0(\mathbf{v}_2) + 0(\mathbf{v}_3) + \cdots + 0(\mathbf{v}_k) = \mathbf{0}.$$

   Since the coefficient of $\mathbf{0}$ (on the left-hand side) is '1', we have a nontrivial vanishing linear combination of the vectors of S. Therefore, S is dependent. ∎

Problem

Let V be a vector space and let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an independent subset of V. Is

$$S = \{\mathbf{u} + \mathbf{v}, 2\mathbf{u} + \mathbf{w}, \mathbf{v} - 5\mathbf{w}\}$$

an independent subset of V? Justify your answer.

Solution

Suppose that a linear combination of the vectors of S is equal to zero, i.e.,

$$a(\mathbf{u} + \mathbf{v}) + b(2\mathbf{u} + \mathbf{w}) + c(\mathbf{v} - 5\mathbf{w}) = 0$$

for some $a, b, c \in \mathbb{R}$. Then $(a + 2b)\mathbf{u} + (a + c)\mathbf{v} + (b - 5c)\mathbf{w} = 0$. Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is independent,

$$\begin{aligned} a + 2b &= 0 \\ a + c &= 0 \\ b - 5c &= 0. \end{aligned}$$

Solving for a, b and c, we find that the system has unique solution $a = b = c = 0$. Therefore, S is linearly independent. ■

Problem

Suppose that A is an $n \times n$ matrix with the property that $A^k = \mathbf{0}$ but $A^{k-1} \neq \mathbf{0}$. Prove that

$$B = \{I, A, A^2, \ldots, A^{k-1}\}$$

is an independent subset of $\mathbf{M}_{nn}$.

Solution (hints only)

Use the standard approach: take a linear combination of the matrices and set it equal to the $n \times n$ zero matrix. Two key points:

- Since $A^{k-1} \neq \mathbf{0}$, the matrices $A, A^2, \ldots, A^{k-2}$ are all nonzero.
- Since $A^k = \mathbf{0}$, the matrices $A^{k+1}, A^{k+2}, A^{k+3}, \ldots$ are all zero.

Theorem (Unique Representation Theorem)

Let V be a vector space and let $\mathrm{U} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathrm{V}$ be an independent set. If $\mathbf{v}$ is in $\mathrm{span}(\mathrm{U})$, then $\mathbf{v}$ has a unique representation as a linear combination of elements of U.

Again, the proof of the corresponding result for $\mathbb{R}^n$ generalizes to an arbitrary vector space V.

The Fundamental Theorem

The Fundamental Theorem for $\mathbb{R}^n$ generalizes to an arbitrary vector space.

Theorem (Fundamental Theorem)

Let V be a vector space that can be spanned by a set of n vectors, and suppose that V contains an independent subset of m vectors. Then $m \leq n$.

Proof.

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and let $Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_m\}$. Suppose $V = \text{span}(X)$ and that Y is an independent subset of V. Each vector in Y can be written as a linear combination of vectors of X: for some $a_{ij} \in \mathbb{R}$, $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\begin{array}{rcl} \mathbf{y}_1 & = & a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n \\ \mathbf{y}_2 & = & a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n \\ \vdots & = & \vdots \\ \mathbf{y}_m & = & a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_n. \end{array}$$

Proof. (continued)

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$, and suppose that $m > n$. Since $\text{rank}\,(A) = \dim(\text{row}(A)) \leq n$, it follows that the rows of A form a dependent subset of $\mathbb{R}^n$, and hence there is a nontrivial linear combination of the rows of A that is equal to the $1 \times n$ vector of all zeros, i.e., there exist $s_1, s_2, \ldots, s_m \in \mathbb{R}$, not all equal to zero, such that

$$\begin{bmatrix} s_1 & s_2 & \cdots & s_m \end{bmatrix} A = \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} = \mathbf{0}_{1n}.$$

It follows that for each j, $1 \leq j \leq n$,

$$s_1 a_{1j} + s_2 a_{2j} + \ldots + s_m a_{mj} = 0. \tag{1}$$

Consider the (nontrivial) linear combination of vectors of Y:

$$s_1 \mathbf{y}_1 + s_2 \mathbf{y}_2 + \cdots + s_m \mathbf{y}_m.$$

Proof. (continued)

$$
\begin{aligned}
s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m &= s_1(a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n) + \\
&\quad s_2(a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n) + \\
&\quad \vdots \\
&\quad s_m(a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_n) \\
&= (s_1a_{11} + s_2a_{21} + \ldots + s_ma_{m1})\mathbf{x}_1 + \\
&\quad (s_1a_{12} + s_2a_{22} + \ldots + s_ma_{m2})\mathbf{x}_2 + \\
&\quad \vdots \\
&\quad (s_1a_{1n} + s_2a_{2n} + \ldots + s_ma_{mn})\mathbf{x}_n.
\end{aligned}
$$

By Equation (1), it follows that

$$s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m = 0\mathbf{x}_1 + 0\mathbf{x}_2 + \cdots + 0\mathbf{x}_n = \mathbf{0}.$$

Therefore, $s_1\mathbf{y}_1 + s_2\mathbf{y}_2 + \cdots + s_m\mathbf{y}_m = \mathbf{0}$ is a nontrivial vanishing linear combination of the vectors of Y. This contradicts the fact that Y is independent, and therefore $m \leq n$. ■

Bases and Dimension

Definition

Let V be a vector space and let $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \subseteq V$. We say B is a basis of V if
(i) B is an independent subset of V and
(ii) $\text{span}(B) = V$.

Remark (Unique Representation Theorem)

Recall that if V is a vector space and B is a basis of V, then as seen earlier, any vector $\mathbf{u} \in V$ can be expressed uniquely as a linear combination of vectors of B.

Example

As we saw earlier, $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is a basis of $\mathbb{R}^n$, called the standard basis of $\mathbb{R}^n$.

Example (A basis of $\mathcal{P}_n$)

We've already seen that

$$\{1, x, x^2, \ldots, x^n\}$$

spans $\mathcal{P}_n$ and is an independent subset of $\mathcal{P}_n$, and is thus a basis of $\mathcal{P}_n$.

$$\{1, x, x^2, \ldots, x^n\}$$

is called the standard basis of $\mathcal{P}_n$.

Example (A basis of $\mathbf{M}_{mn}$)

The set of mn $m \times n$ matrices that have a '1' in position (i, j) and zeros elsewhere, $1 \leq i \leq m$, $1 \leq j \leq n$, spans $\mathbf{M}_{mn}$ and is an independent subset of $\mathbf{M}_{mn}$. Therefore, this set constitutes a basis of $\mathbf{M}_{mn}$ and is called the standard basis of $\mathbf{M}_{mn}$.

The Invariance Theorem generalizes from $\mathbb{R}^n$ to an arbitrary vector space V. The proof is identical, and involves two applications of the Fundamental Theorem.

Theorem (Invariance Theorem)

If V is a vector space with bases $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$, then $m = n$.

Definition (Dimension of a vector space)

Let V be a vector space and suppose $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis of V. The dimension of V is the number of vectors in B, and we write $\dim(V) = n$. By convention, $\dim(\{\mathbf{0}\}) := 0$.

Example

Let V be a vector space and $\mathbf{u}$ a NONZERO vector of V. Then $U = \text{span}\{\mathbf{u}\}$ is spanned by $\{\mathbf{u}\}$. Since $\{\mathbf{u}\}$ is independent, $\{\mathbf{u}\}$ is a basis of U, and thus $\dim(U) = 1$.

Example

Since $\{1, x, x^2, \ldots, x^n\}$ is a basis of $\mathcal{P}_n$, $\dim(\mathcal{P}_n) = n + 1$.

Example

$\dim(\mathbf{M}_{mn}) = mn$ since the standard basis of $\mathbf{M}_{mn}$ consists of mn matrices.

Problem

Let $U = \left\{ A \in \mathbf{M}_{22} \;\middle|\; A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A \right\}$. Then U is a subspace of $\mathbf{M}_{22}$. Find a basis of U, and hence $\dim(U)$.

Solution

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}$. Then

$$A \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}.$$

If $A \in U$, then $\begin{bmatrix} a+b & -b \\ c+d & -d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix}$.

Solution (continued)

Equating entries leads to a system of four equations in the four variables a, b, c and d.

$$\begin{array}{rcl} a+b &=& a+c \\ -b &=& b+d \\ c+d &=& -c \\ -d &=& -d \end{array} \quad \text{or} \quad \begin{array}{rcl} b-c &=& 0 \\ -2b-d &=& 0 \\ 2c+d &=& 0 \end{array}.$$

The solution to this system is $a=s$, $b=-\frac{1}{2}t$, $c=-\frac{1}{2}t$, $d=t$ for any $s,t\in\mathbb{R}$, and thus $A=\begin{bmatrix} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{bmatrix}$, $s,t\in\mathbb{R}$. Since $A\in U$ is arbitrary,

$$\begin{aligned} U &= \left\{\begin{bmatrix} s & \frac{t}{2} \\ -\frac{t}{2} & t \end{bmatrix} \middle| \; s,t\in\mathbb{R}\right\} \\ &= \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \middle| \; s,t\in\mathbb{R}\right\} \\ &= \operatorname{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\right\}. \end{aligned}$$

Solution (continued)

Let

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \right\}.$$

Then $\mathrm{span}(B) = U$, and it is routine to verify that B is an independent subset of $\mathbf{M}_{22}$. Therefore B is a basis of $\mathbf{M}_{22}$, and $\dim(U) = 2$. ■

Problem

Let $U = \{p(x) \in \mathcal{P}_2 \mid p(1) = 0\}$. Then U is a subspace of $\mathcal{P}_2$. Find a basis of U, and hence $\dim(U)$.

Solution

Final Answer $B = \{x - x^2, 1 - x^2\}$ is a basis of U and thus $\dim(U) = 2$. ■