Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-4. Finite Dimensional Spaces

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Generalizing from $\mathbb{R}^n$

Constructing basis from independent sets by adding vectors

Subspaces of finite dimensional vector spaces

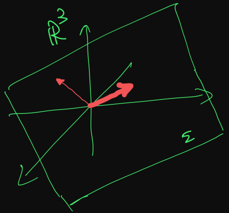
Constructing basis from spanning sets by deleting vectors

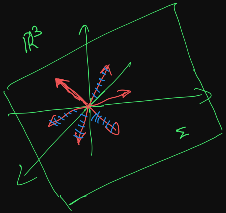
Sums and Intersections

Generalizing from $\mathbb{R}^n$

We have learnt that for a subspace U of $\mathbb{R}^n$, if $U \neq \{\mathbf{0}\}$, then

1. U has a basis, and $\dim(U) \leq n$.
2. Any independent subset of U can be extended (by adding vectors) to a basis of U.
3. Any spanning set of U can be cut down (by deleting vectors) to a basis of U.





Definition

A vector space V is finite dimensional if it is spanned by a finite set of vectors. Otherwise it is called infinite dimensional.

Example

1. $\mathbb{R}^n$, $\mathcal{P}_n$ and $\mathbf{M}_{mn}$ are all examples of finite dimensional vector spaces
2. The zero vector space, $\{\mathbf{0}\}$, is also finite dimensional, since it is spanned by $\{\mathbf{0}\}$.
3. $\mathcal{P}$ is an infinite dimensional vector space.

Lemma (Independent Lemma)

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ an independent subset of V. Suppose $\mathbf{u}$ is a vector in V. Then

$$\mathbf{u} \notin \operatorname{span}(S) \implies S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\} \text{ is independent.}$$

Proof.

Suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k + a\mathbf{u} = \mathbf{0}$. We claim that $a = 0$. Otherwise, if $a \neq 0$, then

$$a\mathbf{u} = -a_1\mathbf{v}_1 - a_2\mathbf{v}_2 - \cdots - a_k\mathbf{v}_k,$$

implying that

$$\mathbf{u} = -\frac{a_1}{a}\mathbf{v}_1 - \frac{a_2}{a}\mathbf{v}_2 - \cdots - \frac{a_k}{a}\mathbf{v}_k,$$

i.e., $\mathbf{u} \in \operatorname{span}(S)$, a contradiction. Therefore, $a = 0$.
Now $a = 0$ implies that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$. Since S is independent, $a_1 = a_2 = \cdots = a_k = 0$, and it follows that S' is independent. ■

Remark

Under the setting of the Independent Lemma, for $\mathbf{u} \in V$, we have indeed:

$$\mathbf{u} \notin \operatorname{span}(S) \iff S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{u}\} \text{ is independent.}$$

Lemma

Let V be a finite dimensional vector space. If U is any subspace of V, then any independent subset of U can be extended to a finite basis of U.

Algorithm 1: Proof of Lemma

Input : 1. V: finite dimensional vector space
2. $U \subseteq V$ a subspace
3. $W_0 \subseteq U$ an independent subset of U

$W_0 \rightarrow W$;
while $\text{span}\{W\} \neq U$ **do**
  Pick up arbitrary $\mathbf{x} \in U \setminus \text{span}\{W\}$;
  $\{\mathbf{x}\} \cup W \rightarrow W$;
  Independent Lemma guarantees that the new W is an independent set;
end

Output: W, that is independent and spans U; hence a basis of U.

Constructing basis from independent sets by adding vectors

Theorem

Let V be a finite dimensional vector space spanned by a set of m vectors.

(1) V has a finite basis, and $\dim(V) \leq m$.

(2) Every independent subset of V can be extended to a basis of V by adding vectors from any fixed basis of V.

(3) If U is a subspace of V, then
(i) U is finite dimensional and $\dim(U) \leq \dim(V)$;
(ii) every basis of U is part of a basis of V.

Proof.

(1) If $V = \{\mathbf{0}\}$, then V has dimension zero, and the (unique) basis of V is the empty set. Otherwise, choose any nonzero vector $\mathbf{x}$ in V and extend $\{\mathbf{x}\}$ to a finite basis B of V (by a previous Lemma). By the Fundamental Theorem, B has at most m elements, so $\dim(V) \leq m$.

Proof.

(2)

Algorithm 2: Proof of part 2

Input : 1. V: finite dimensional vector space spanned by m vectors
2. B: a basis of V (exists by part (1))
3. W_0: an independent set of vectors in V

$W_0 \to W$;
while $\text{span}\{W\} \neq V$ **do**
  Find out one $\mathbf{x} \in B \setminus \text{span}\{W\}$;
  $\{\mathbf{x}\} \cup W \to W$;
  Independent Lemma guarantees that the new W is an independent set;
end

Output: W, that is independent and spans V; hence a basis of V.

Proof.

(3-i) If $U = \{\mathbf{0}\}$, then $\dim(U) = 0 \leq m = \dim(V)$. Otherwise, choose $\mathbf{x}$ to be any nonzero vector of U and extend $\{\mathbf{x}\}$ to a basis B of U (again by a previous Lemma). Since B is an independent subset of V, B has at most $\dim(V)$ elements, so $\dim(U) \leq \dim(V)$.

(3-ii) If $U = \{\mathbf{0}\}$, then any basis of V suffices. Otherwise, any basis B of U can be extended to a basis of V: because B is independent, we apply part (2) of this theorem. ∎

Problem

Extend the independent set $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^4$.

Solution (method 1.)

Let $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 3 & 4 & 5 \end{bmatrix}$. Because the elementary row operations won't change row space, let's find the reduced row-echelon form of A

$$R = \begin{bmatrix} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \end{bmatrix}.$$

(row(A) = row(R).) We need add two rows to R to get a nonsingular matrix:

$$\begin{bmatrix} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Solution (continued)

There are certainly multiple choices for those two rows. The simplest choice might be the following:

$$\left[\begin{array}{cc|cc} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence,

$$\mathrm{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{e}_3, \vec{e}_4 \right\},$$

gives a basis for $\mathbb{R}^4$. ■

Below is a more systematical way to find all possible choices based on one basis from V

Solution (method 2.)

$$A = \left[\begin{array}{cc|cccc} 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ -1 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \to R = \left[\begin{array}{cc|cccc} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{7}{5} & -\frac{2}{5} & 1 & 0 \\ 0 & 0 & -\frac{2}{5} & -\frac{7}{5} & 0 & 1 \end{array}\right]$$

Now we need to find four columns which include the first two columns from the six columns of R to form a nonsingular matrix. Then the corresponding columns from A form a basis for $\mathbb{R}^4$. Indeed, we can choose any two columns from the last four columns. If we choose the last two columns, this will give the result from the previous answer. ■

Problem

Extend the independent set $S = \{x^2 - 3x + 1, 2x^3 + 3\}$ to a basis of $\mathcal{P}_3$.

Solution (method 1.)

Using the fact that polynomials of distinct orders are independent, we need only include missing orders. Hence: $B = \{1, x, x^2 - 3x + 1, 2x^3 + 3\}$. ∎

Remark

What happens if $S = \{x^2 - 3x + 1, 2x^2 + 3\}$?

Solution (method 2.)

Transform each vector – polynomial – to a row vector and form a matrix:

$$A = \begin{pmatrix} 1 & -3 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{pmatrix}$$

Now the question is how one can add two rows to A to make it nonsingular:

$$\begin{pmatrix} 1 & -3 & 1 & 0 \\ 3 & 0 & 0 & 2 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

It is ready to check that the last two rows to be any of the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \ldots$$

For example, if we choose make the first choice, this will give us $\{1, x\}$ as the additional two polynomials. Therefore, we obtain a basis:
$B = \{1, x, x^2 - 3x + 1, 2x^3 + 3\}$. ■

Solution (method 3.)

Carry out columns-wise... ■

Problem

Extend the independent set

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

to a basis of $\mathbf{M}_{22}$.

Solution

S can be extended to a basis of $\mathbf{M}_{22}$ by adding a matrix from the standard basis of $\mathbf{M}_{22}$. To methodically find such a matrix, try to express each matrix of the standard basis of $\mathbf{M}_{22}$ as a linear combination of the matrices of S. This results in four systems of linear equations, each in three variables, and these can be solved simultaneously by putting the augmented matrix in row-echelon form.

$$\left[\begin{array}{rrr|rrrr} -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rrrr} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right].$$

Solution (continued)

The row-echelon matrix indicates that all four systems are inconsistent, and thus any of the four matrices in the standard basis of $\mathbf{M}_{22}$ can be used to extend S to an independent subset of four vectors (matrices) of $\mathbf{M}_{22}$. Let

$$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

If $\operatorname{span}(B) \neq \mathbf{M}_{22}$, then apply the Independent Lemma to get an independent set with five vectors (matrices). Since $\mathbf{M}_{22}$ is spanned by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

this contradicts the Fundamental Theorem. Therefore $\operatorname{span}(B) = \mathbf{M}_{22}$, and B is a basis of $\mathbf{M}_{22}$. $\blacksquare$

Subspaces of finite dimensional vector spaces

Theorem

Let V be a finite dimensional vector space, and let U and W be subspaces of V.

1. If $U \subseteq W$, then $\dim(U) \leq \dim(W)$.
2. If $U \subseteq W$ and $\dim(U) = \dim(W)$, then $U = W$.

This is the generalization to finite dimensional vector spaces of the corresponding result for $\mathbb{R}^n$.

Proof.

1. Since W is a subspace of a finite dimensional vector space, this result follows from a previous Theorem.
2. Let B be a basis of U, and suppose $|B| = k = \dim(W)$. Since $U \subseteq W$, B is an independent subset of W. If $\mathrm{span}(B) \neq W$, then W contains an independent set of size $k + 1$, contradicting the Fundamental Theorem. Therefore, B is a basis of W, and thus $U = W$.

$\blacksquare$

Problem

Let $a \in \mathbb{R}$ be fixed, and let

$$U = \{p(x) \in \mathcal{P}_n \mid p(a) = 0\}.$$

Then U is a subspace of $\mathcal{P}_n$ (you should be able to prove this). Show that

$$S = \{(x-a), (x-a)^2, (x-a)^3, \ldots, (x-a)^n\}$$

is a basis of U.

Remark (Hints of the proof)

We need to show that the following:

1. Show that $\operatorname{span}(S) \subseteq U$, and that S is independent.
2. Deduce that $n \leq \dim(U) \leq n+1$.
3. Show that $\dim(U)$ can not equal $n+1$.

Solution

- Each polynomial in S has a as a root, so $S \subseteq U$. Since U is a subspace of $\mathcal{P}_n$ it follows that $\mathrm{span}(S) \subseteq U$.
- Since the polynomials in S have distinct degrees ($(x-a)^i$ has degree i), S is independent.
- Since $\mathrm{span}(S) \subseteq U \subseteq \mathcal{P}_n$, it follows that

  $$\dim(\mathrm{span}(S)) \leq \dim(U) \leq \dim(\mathcal{P}_n).$$

  Since S is a basis of $\mathrm{span}(S)$, $\dim(\mathrm{span}(S)) = n$; also, $\dim(\mathcal{P}_n) = n+1$, and thus $n \leq \dim(U) \leq n+1$.
- Finally, if $\dim(U) = n+1$, then $U = \mathcal{P}_n$, implying that every polynomial in $\mathcal{P}_n$ has a as a root. However, $x - a + 1 \in \mathcal{P}_n$ but $x - a + 1 \notin U$, so $\dim(U) \neq n+1$. Therefore, $\dim(U) = n$.

We now have $\mathrm{span}(S) \subseteq U$ and $\dim(\mathrm{span}(S)) = n = \dim(U)$. By a previous Theorem, $U = \mathrm{span}(S)$, and hence S is a basis of U. $\blacksquare$

Lemma (Dependent Lemma)

Let V be a vector space and $D = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ a subset of V, $k \geq 2$. Then D is dependent if and only if there is some vector in D that is a linear combination of the other vectors in D.

Proof.

"$\Rightarrow$" Suppose that D is dependent. Then

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

for some $t_1, t_2, \ldots, t_k \in \mathbb{R}$ not all equal to zero. Note that we may assume that $t_1 \neq 0$. Then

$$\begin{aligned} t_1\mathbf{v}_1 &= -t_2\mathbf{v}_2 - t_3\mathbf{v}_3 - \cdots - t_k\mathbf{v}_k \\ \mathbf{v}_1 &= -\frac{t_2}{t_1}\mathbf{v}_2 - \frac{t_3}{t_1}\mathbf{v}_3 - \cdots - \frac{t_k}{t_1}\mathbf{v}_k; \end{aligned}$$

i.e., $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$.

Proof. (continued)

“$\Leftarrow$" Conversely, assume that some vector in D is a linear combination of the other vectors of D. We may assume that $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k$. Then

$$\mathbf{v}_1 = s_2\mathbf{v}_2 + s_3\mathbf{v}_3 + \cdots + s_k\mathbf{v}_k$$

for some $s_2, s_3, \ldots, s_k \in \mathbb{R}$, implying that

$$1\mathbf{v}_1 - s_2\mathbf{v}_2 - s_3\mathbf{v}_3 - \cdots - x_k\mathbf{v}_k = \mathbf{0}.$$

Thus there is a nontrivial linear combination of the vectors of D that vanishes, so D is dependent. ■

Suppose $U = \text{span}(S)$ for some set of vectors S. If S is dependent, then we can find a vector $\mathbf{v}$ in S that is a linear combination of the other vectors of S. Deleting $\mathbf{v}$ from S results if a set T with $\text{span}(T) = \text{span}(S) = U$.

Constructing basis from spanning sets by deleting vectors

Theorem

Let V be a finite dimensional vector space. Then any spanning set S of V can be cut down to a basis of V by deleting vectors of S.

Proof.

Algorithm 3: Proof of Theorem

Input : 1. V: finite dimensional vector space spanned by m vectors
3. S: a spanning set of V

$S \to W$;
while W is dependent do
  Find out one $\mathbf{x} \in W$ that can be linearly represented by the rest;
  $W \setminus \{\mathbf{x}\} \to W$;
  Dependent Lemma guarantees that the span of the new W remains to be V;
end
Output: W, that is independent and spans V; hence a basis of V.

■

Problem

Let

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad X_5 = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix},$$

and let $U = \{X_1, X_2, X_3, X_4, X_5\}$. Then $\text{span}(U) = \mathbf{M}_{22}$. Find a basis of $\mathbf{M}_{22}$ from among the elements of U.

Solution

Since U has five matrices and $\dim(\mathbf{M}_{22}) = 4$, U is dependent. Suppose

$$aX_1 + bX_2 + cX_3 + dX_4 + eX_5 = \mathbf{0}_{22}.$$

This gives us a homogeneous system of four equations in five variables, whose general solution is

$$a = -\frac{4}{3}t; \quad b = \frac{1}{3}t; \quad c = -\frac{2}{3}t; \quad d = 0; \quad e = t, \qquad \text{for } t \in \mathbb{R}.$$

Solution (continued)

Taking $t = 3$ gives us

$$-4X_1 + X_2 - 2X_3 + 3X_5 = \mathbf{0}_{22}.$$

From this, we see that X_1 can be expressed as a linear combination of X_2, X_3 and X_5.

Let

$$B = \{X_2, X_3, X_4, X_5\}.$$

Then $\text{span}(B) = \text{span}(U) = \mathbf{M}_{22}$. If B is not independent, then apply the Dependent Lemma to find a subset of three matrices of B that spans $\mathbf{M}_{22}$. Since

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is an independent subset of $\mathbf{M}_{22}$, this contradicts the Fundamental Theorem. Therefore B is independent, and hence is a basis of $\mathbf{M}_{22}$. ■

Theorem (Generalization of $\mathbb{R}^n$)

Let V be a finite dimensional vector space with $\dim(V) = n$, and suppose S is a subset of V containing n vectors. Then S is independent if and only if S spans V.

Proof.

$(\Rightarrow)$ Suppose S is independent. Since every independent set of V can be extended to a basis of V, there exists a basis B of V with $S \subseteq B$. However, $|S| = n$ and $|B| = n$, and therefore $S = B$, i.e., S is a basis of V. In particular, this implies that S spans V.

$(\Leftarrow)$ Conversely, suppose that $\operatorname{span}(S) = V$. Since every spanning set of V can be cut down to a basis of V, there exists a basis B of V with $B \subseteq S$. However, $|S| = n$ and $|B| = n$, and therefore $S = B$, i.e., S is a basis of V. In particular, this implies that S is an independent set of V. $\blacksquare$

Remark

This theorem can be used to simplify the arguments used in various problems covered.

Problem

Find a basis of $\mathcal{P}_2$ among the elements of the set

$$U = \left\{x^2 - 3x + 2, \quad 1 - 2x, \quad 2x^2 + 1, \quad 2x^2 - x - 3\right\}.$$

Solution

Since $|U| = 4 > 3 = \dim(\mathcal{P}_2)$, U is dependent.

Suppose $a(x^2 - 3x + 2) + b(1 - 2x) + c(2x^2 + 1) + d(2x^2 - x - 3) = 0$; then

$$(a + 2c + 2d)x^2 + (-3a - 2b - d)x + (2a + b + c - 3d) = 0.$$

This leads to a system of three equations in four variables that can be solved using gaussian elimination.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 2 & 2 & 0 \\ -3 & -2 & 0 & -1 & 0 \\ 2 & 1 & 1 & -3 & 0 \end{array}\right] \to \left[\begin{array}{rrrr|r} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Thus $a = -2t$, $b = 3t$, $c = t$ and $d = 0$ for any $t \in \mathbb{R}$. Also, since each row of the reduced row-echelon matrix has a leading one, U spans $\mathcal{P}_2$.

Solution (continued)

Let $t = -1$. Then

$$2(x^2 - 3x + 2) - 3(1 - 2x) - (2x^2 + 1) = 0,$$

so any one of $\{x^2 - 3x + 2, 1 - 2x, 2x^2 + 1\}$ can be expressed as a linear combination of the other two. Let's remove $x^2 - 3x + 2$. Hence, set

$$B = \{1 - 2x, 2x^2 + 1, 2x^2 - x - 3\}.$$

Then $\text{span}(B) = \text{span}(U) = \mathcal{P}_2$. Since $|B| = 3 = \dim(\mathcal{P}_2)$, it follows from that B is independent. Therefore, $B \subseteq U$ is a basis of $\mathcal{P}_2$. ■

Problem

Let $V = \{A \in \mathbf{M}_{22} \mid A^T = A\}$. Then V is a vector space. Find a basis of V consisting of invertible matrices.

Remark

Note that V is the set of 2×2 symmetric matrices, so

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

From this, we deduce that $\dim(V) = 3$. (Why?) Thus, a basis of V consisting of invertible matrices will consist of three independent symmetric invertible matrices.

Solution

There are many solutions. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix B is invertible, so one approach is to take linear combinations of A and C to produce two independent invertible matrices; for example

$$A + C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A - C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to verify that $S = \{A + C, A - C, B\}$ is an independent subset of 2×2 invertible symmetric matrices. Since $|S| = 3 = \dim(V)$, S spans V and is therefore a basis of V. ■

Sums and Intersections

Definition

Let V be a vector space, and let U and W be subspaces of V. Then

1. $\mathrm{U} + \mathrm{W} = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in \mathrm{U} \quad \text{and} \quad \mathbf{w} \in \mathrm{W}\}$ and is called the sum of U and W.
2. $\mathrm{U} \cap \mathrm{W} = \{\mathbf{v} \mid \mathbf{v} \in \mathrm{U} \quad \text{and} \quad \mathbf{v} \in \mathrm{W}\}$ and is called the intersection of U and W.
3. If U and W are subspaces of a vector space V and $\mathrm{U} \cap \mathrm{W} = \{\mathbf{0}\}$, then the sum of U and W is call the direct sum and is denoted $\mathrm{U} \oplus \mathrm{W}$.

Lemma

Prove that both $\mathrm{U} + \mathrm{W}$ and $\mathrm{U} \cap \mathrm{W}$ are subspaces of V.

Proof. (of $U + W$)

1. Since U and W are subspaces of V, $\mathbf{0}$, the zero vector of V, is an element of both U and W. Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, $\mathbf{0} \in U + W$.
2. Let $\mathbf{x}_1, \mathbf{x}_2 \in U + W$. Then $\mathbf{x}_1 = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{x}_2 = \mathbf{u}_2 + \mathbf{w}_2$ for some $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$. It follows that

   $$\mathbf{x}_1 + \mathbf{x}_2 = (\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2).$$

   Since U and W are subspaces of V, $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$, and therefore $\mathbf{x}_1 + \mathbf{x}_2 \in U + W$.
3. Let $\mathbf{x}_1 \in U + W$ and $k \in \mathbb{R}$. Then $\mathbf{x}_1 = \mathbf{u}_1 + \mathbf{w}_1$ for some $\mathbf{u}_1 \in U$ and $\mathbf{w}_1 \in W$. It follows that $k\mathbf{x}_1 = k(\mathbf{u}_1 + \mathbf{w}_1) = (k\mathbf{u}_1) + (k\mathbf{w}_1)$. Since U and W are subspaces of V, $k\mathbf{u}_1 \in U$ and $k\mathbf{w}_1 \in W$, and therefore $k\mathbf{x}_1 \in U + W$.

By the Subspace Test, $U + W$ is a subspace of V. ■

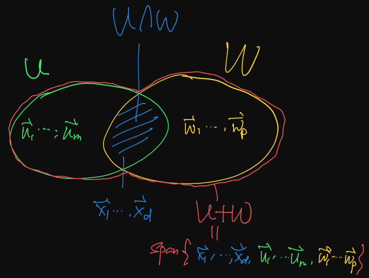
Theorem

If U and W are finite dimensional subspaces of a vector space V, then U + W is finite dimensional and

$$\dim(\mathrm{U} + \mathrm{W}) = \dim(\mathrm{U}) + \dim(\mathrm{W}) - \dim(\mathrm{U} \cap \mathrm{W}).$$

Remark

V need not be finite dimensional!



Proof.

$U \cap W$ is a subspace of the finite dimensional vector space U, so is finite dimensional, and has a finite basis $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d\}$. Since $X \subseteq U \cap W$, X can be extended to a finite basis B_U of U and a finite basis B_W of W:

$$B_U = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \quad \text{and} \quad B_W = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}.$$

Then

$$\operatorname{span}\{\mathbf{x}_1, \cdots \mathbf{x}_d, \mathbf{u}_1, \cdots, \mathbf{u}_m, \mathbf{w}_1, \cdots, \mathbf{w}_p\} = U + W.$$

Proof. (continued)

What remains is to prove that

$$B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$$

is a basis of $U + W$ since then it implies that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\Updownarrow$$

$$d + m + p = (d + m) + (d + p) - d$$

Proof. (continued)

To prove B is linearly independent, we need to show that

$$r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m + t_1\mathbf{w}_1 + \cdots + t_p\mathbf{w}_p = \mathbf{0}.$$

which is equivalent to

$$\underbrace{r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d + s_1\mathbf{u}_1 + \cdots + s_m\mathbf{u}_m}_{\in U} = \underbrace{-t_1\mathbf{w}_1 - \cdots - t_p\mathbf{w}_p}_{\in W}$$

Hence,

1. LHS $\in U \cap W$, which implies that $s_1 = \cdots = s_m = 0$.
2. RHS $\in U \cap W$, which implies that $t_1 = \cdots = t_p = 0$.

Finally,

$$r_1\mathbf{x}_1 + \cdots + r_d\mathbf{x}_d = \mathbf{0}$$

implies that $r_1 = \cdots = r_d = 0$. This proves that B is independent. ■