

Math 221: LINEAR ALGEBRA

Chapter 6. Vector Spaces

§6-4. Finite Dimensional Spaces

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¹Slides are adapted from those by Karen Seyffarth from University of Calgary.

Generalizing from $\mathbb{R}^n$

Constructing basis from independent sets by adding vectors

Subspaces of finite dimensional vector spaces

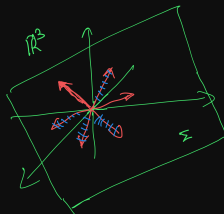
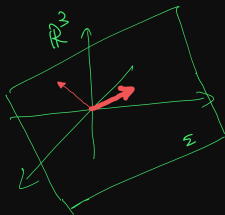
Constructing basis from spanning sets by deleting vectors

Sums and Intersections

Generalizing from $\mathbb{R}^n$

We have learnt that for a subspace U of $\mathbb{R}^n$, if $U \neq \{\mathbf{0}\}$, then

1. U has a basis, and $\dim(U) \leq n$.
2. Any **independent subset** of U can be extended (by **adding** vectors) to a basis of U .
3. Any **spanning set** of U can be cut down (by **deleting** vectors) to a basis of U .



Definition

A vector space V is **finite dimensional** if it is spanned by a finite set of vectors. Otherwise it is called **infinite dimensional**.

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Example

1. $\mathbb{R}^n$, $\mathcal{P}_n$ and $\mathbf{M}_{mn}$ are all examples of finite dimensional vector spaces
2. The zero vector space, $\{\mathbf{0}\}$, is also finite dimensional, since it is spanned by $\{\mathbf{0}\}$.
3. $\mathcal{P}$ is an infinite dimensional vector space.

Lemma (Independent Lemma)

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ an independent subset of V . Suppose $\mathbf{u}$ is a vector in V . Then

$$\mathbf{u} \notin \text{span}(S) \implies S' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}\} \text{ is independent.}$$

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Proof.

Suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k + a\mathbf{u} = \mathbf{0}$.

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Proof.

Suppose that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k + a\mathbf{u} = \mathbf{0}$. We claim that $a = 0$. Otherwise, if $a \neq 0$, then

$$a\mathbf{u} = -a_1\mathbf{v}_1 - a_2\mathbf{v}_2 - \dots - a_k\mathbf{v}_k,$$

implying that

$$\mathbf{u} = -\frac{a_1}{a}\mathbf{v}_1 - \frac{a_2}{a}\mathbf{v}_2 - \dots - \frac{a_k}{a}\mathbf{v}_k,$$

i.e., $\mathbf{u} \in \text{span}(S)$, a contradiction. Therefore, $a = 0$.

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i.e., $\mathbf{u} \in \text{span}(S)$, a contradiction. Therefore, $a = 0$.

Now $a = 0$ implies that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_k\mathbf{v}_k = \mathbf{0}$. Since S is independent, $a_1 = a_2 = \dots = a_k = 0$, and it follows that S' is independent. ■

Remark

Under the setting of the Independent Lemma, for $\mathbf{u} \in V$, we have indeed:

$$\mathbf{u} \notin \text{span}(S) \iff S' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}\} \text{ is independent.}$$

Lemma

Let V be a finite dimensional vector space. If U is any subspace of V , then any independent subset of U can be extended to a finite basis of U .

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Algorithm 2: Proof of Lemma

Input : 1. V : finite dimensional vector space
2. $U \subseteq V$ a subspace
3. $W_0 \subseteq U$ an independent subset of U

$W_0 \rightarrow W$;

while $\text{span}\{W\} \neq U$ do

 Pick up arbitrary $\mathbf{x} \in U \setminus \text{span}\{W\}$;

$\{\mathbf{x}\} \cup W \rightarrow W$;

 Independent Lemma guarantees that the new W is an independent set;

end

Output: W , that is independent and spans U ; hence a basis of U .

Constructing basis from independent sets by adding vectors

Theorem

Let V be a finite dimensional vector space spanned by a set of m vectors.

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- (1) V has a finite basis, and $\dim(V) \leq m$.
- (2) Every independent subset of V can be extended to a basis of V by adding vectors from any fixed basis of V .
- (3) If U is a subspace of V , then
 - (i) U is finite dimensional and $\dim(U) \leq \dim(V)$;
 - (ii) every basis of U is part of a basis of V .

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- (1) V has a finite basis, and $\dim(V) \leq m$.
- (2) Every independent subset of V can be extended to a basis of V by adding vectors from any fixed basis of V .
- (3) If U is a subspace of V , then
 - (i) U is finite dimensional and $\dim(U) \leq \dim(V)$;
 - (ii) every basis of U is part of a basis of V .

Proof.

- (1) If $V = \{\mathbf{0}\}$, then V has dimension zero, and the (unique) basis of V is the empty set. Otherwise, choose any nonzero vector $\mathbf{x}$ in V and extend $\{\mathbf{x}\}$ to a finite basis B of V (by a previous Lemma). By the Fundamental Theorem, B has at most m elements, so $\dim(V) \leq m$.

Proof.

(2)

Algorithm 3: Proof of part 2

Input : 1. V : finite dimensional vector space spanned by m vectors
2. B : a basis of V (exists by part (1))
3. W_0 : an independent set of vectors in V

$W_0 \rightarrow W$;

while $\text{span}\{W\} \neq V$ do

 Find out one $\mathbf{x} \in B \setminus \text{span}\{W\}$;

$\{\mathbf{x}\} \cup W \rightarrow W$;

 Independent Lemma guarantees that the new W is an independent set;

end

Output: W , that is independent and spans V ; hence a basis of V .

Proof.

(3-i) If $U = \{\mathbf{0}\}$, then $\dim(U) = 0 \leq m = \dim(V)$. Otherwise, choose $\mathbf{x}$ to be any nonzero vector of U and extend $\{\mathbf{x}\}$ to a basis B of U (again by a previous Lemma). Since B is an independent subset of V , B has at most $\dim(V)$ elements, so $\dim(U) \leq \dim(V)$.

Proof.

(3-i) If $U = \{\mathbf{0}\}$, then $\dim(U) = 0 \leq m = \dim(V)$. Otherwise, choose $\mathbf{x}$ to be any nonzero vector of U and extend $\{\mathbf{x}\}$ to a basis B of U (again by a previous Lemma). Since B is an independent subset of V , B has at most $\dim(V)$ elements, so $\dim(U) \leq \dim(V)$.

(3-ii) If $U = \{\mathbf{0}\}$, then any basis of V suffices. Otherwise, any basis B of U can be extended to a basis of V : because B is independent, we apply part (2) of this theorem. 

Problem

Extend the independent set $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ to a basis of $\mathbb{R}^4$.

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Solution (method 1.)

Let $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 2 & 3 & 4 & 5 \end{bmatrix}$. Because the elementary row operations won't change row space, let's find the reduced row-echelon form of A

$$R = \begin{bmatrix} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \end{bmatrix}.$$

($\text{row}(A) = \text{row}(R)$.) We need add two rows to R to get a nonsingular matrix:

$$\begin{bmatrix} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$$

Solution (continued)

There are certainly multiple choices for those two rows. The simplest choice might be the following:

$$\left[\begin{array}{cc|cc} 1 & 0 & 7/5 & 2/5 \\ 0 & 1 & 2/5 & 7/5 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Hence,

$$B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{e}_3, \vec{e}_4 \right\},$$

gives a basis for $\mathbb{R}^4$.



Below is a more systematical way to find all possible choices based on one basis from V

Solution (method 2.)

$$A = \left[\begin{array}{cc|cccc} 1 & 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 & 1 & 0 \\ -1 & 5 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow R = \left[\begin{array}{cc|cccc} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 &; -\frac{3}{5} & \frac{2}{5} & 1 & 0 \\ 0 & 0 & -\frac{3}{5} & -\frac{2}{5} & 0 & 1 \end{array} \right]$$

Now we need to find four columns which include the first two columns from the six columns of R to form a nonsingular matrix. Then the corresponding columns from A form a basis for $\mathbb{R}^4$. Indeed, we can choose any two columns from the last four columns. If we choose the last two columns, this will give the result from the previous answer. ■

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Extend the independent set $S = \{x^2 - 3x + 1, 2x^3 + 3\}$ to a basis of $\mathcal{P}_3$.

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Solution (method 1.)

Using the fact that polynomials of distinct orders are independent, we need only include missing orders. Hence: $B = \{1, x, x^2 - 3x + 1, 2x^3 + 3\}$. ■

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Remark

What happens if $S = \{x^2 - 3x + 1, 2x^2 + 3\}$?

Solution (method 2.)

Transform each vector – polynomial – to a row vector and form a matrix:

$$A = \begin{pmatrix} 1 & -3 & 1 & 0 \\ 3 & 0 & 0 & 2 \end{pmatrix}$$

Now the question is how one can add two rows to A to make it nonsingular:

$$\begin{pmatrix} 1 & -3 & 1 & 0 \\ 3 & 0 & 0 & 2 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

It is ready to check that the last two rows to be any of the following:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \dots$$

For example, if we choose make the first choice, this will give us $\{1, x\}$ as the additional two polynomials. Therefore, we obtain a basis:

$$B = \{1, x, x^2 - 3x + 1, 2x^3 + 3\}.$$



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Solution (method 3.)

Carry out columns-wise...



Problem

Extend the independent set

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

to a basis of $\mathbf{M}_{22}$.

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to a basis of $\mathbf{M}_{22}$.

Solution

S can be extended to a basis of $\mathbf{M}_{22}$ by adding a matrix from the standard basis of $\mathbf{M}_{22}$. To methodically find such a matrix, try to express each matrix of the standard basis of $\mathbf{M}_{22}$ as a linear combination of the matrices of S . This results in four systems of linear equations, each in three variables, and these can be solved simultaneously by putting the augmented matrix in row-echelon form.

$$\left[\begin{array}{ccc|cccc} -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|cccc} 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{array} \right].$$

Solution (continued)

The row-echelon matrix indicates that all four systems are inconsistent, and thus any of the four matrices in the standard basis of $\mathbf{M}_{22}$ can be used to extend S to an independent subset of four vectors (matrices) of $\mathbf{M}_{22}$. Let

$$B = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

If $\text{span}(B) \neq \mathbf{M}_{22}$, then apply the Independent Lemma to get an independent set with five vectors (matrices). Since $\mathbf{M}_{22}$ is spanned by

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

this contradicts the Fundamental Theorem. Therefore $\text{span}(B) = \mathbf{M}_{22}$, and B is a basis of $\mathbf{M}_{22}$. 

Subspaces of finite dimensional vector spaces

Theorem

Let V be a finite dimensional vector space, and let U and W be subspaces of V .

1. If $U \subseteq W$, then $\dim(U) \leq \dim(W)$.
2. If $U \subseteq W$ and $\dim(U) = \dim(W)$, then $U = W$.

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Proof.

1. Since W is a subspace of a finite dimensional vector space, this result follows from a previous Theorem.
2. Let B be a basis of U , and suppose $|B| = k = \dim(W)$. Since $U \subseteq W$, B is an independent subset of W . If $\text{span}(B) \neq W$, then W contains an independent set of size $k + 1$, contradicting the Fundamental Theorem. Therefore, B is a basis of W , and thus $U = W$.



Problem

Let $a \in \mathbb{R}$ be fixed, and let

$$U = \{p(x) \in \mathcal{P}_n \mid p(a) = 0\}.$$

Then U is a subspace of $\mathcal{P}_n$ (you should be able to prove this). Show that

$$S = \{(x - a), (x - a)^2, (x - a)^3, \dots, (x - a)^n\}$$

is a basis of U .

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Remark (Hints of the proof)

We need to show that the following:

1. Show that $\text{span}(S) \subseteq U$, and that S is independent.
2. Deduce that $n \leq \dim(U) \leq n + 1$.
3. Show that $\dim(U)$ can not equal $n + 1$.

Solution

- ▶ Each polynomial in S has a as a root, so $S \subseteq U$. Since U is a subspace of $\mathcal{P}_n$ it follows that $\text{span}(S) \subseteq U$.
- ▶ Since the polynomials in S have distinct degrees ($(x - a)^i$ has degree i), S is independent.
- ▶ Since $\text{span}(S) \subseteq U \subseteq \mathcal{P}_n$, it follows that

$$\dim(\text{span}(S)) \leq \dim(U) \leq \dim(\mathcal{P}_n).$$

Since S is a basis of $\text{span}(S)$, $\dim(\text{span}(S)) = n$; also, $\dim(\mathcal{P}_n) = n + 1$, and thus $n \leq \dim(U) \leq n + 1$.

- ▶ Finally, if $\dim(U) = n + 1$, then $U = \mathcal{P}_n$, implying that every polynomial in $\mathcal{P}_n$ has a as a root. However, $x - a + 1 \in \mathcal{P}_n$ but $x - a + 1 \notin U$, so $\dim(U) \neq n + 1$. Therefore, $\dim(U) = n$.

We now have $\text{span}(S) \subseteq U$ and $\dim(\text{span}(S)) = n = \dim(U)$. By a previous Theorem, $U = \text{span}(S)$, and hence S is a basis of U . ■

Lemma (Dependent Lemma)

Let V be a vector space and $D = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ a subset of V , $k \geq 2$.

Then D is dependent if and only if there is some vector in D that is a linear combination of the other vectors in D .

Lemma (Dependent Lemma)

Let V be a vector space and $D = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ a subset of V , $k \geq 2$. Then D is dependent if and only if there is some vector in D that is a linear combination of the other vectors in D .

Proof.

" $\Rightarrow$ " Suppose that D is dependent. Then

$$t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \cdots + t_k \mathbf{v}_k = \mathbf{0}$$

for some $t_1, t_2, \dots, t_k \in \mathbb{R}$ not all equal to zero. Note that we may assume that $t_1 \neq 0$. Then

$$\begin{aligned} t_1 \mathbf{v}_1 &= -t_2 \mathbf{v}_2 - t_3 \mathbf{v}_3 - \cdots - t_k \mathbf{v}_k \\ \mathbf{v}_1 &= -\frac{t_2}{t_1} \mathbf{v}_2 - \frac{t_3}{t_1} \mathbf{v}_3 - \cdots - \frac{t_k}{t_1} \mathbf{v}_k; \end{aligned}$$

i.e., $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_k$.

Proof. (continued)

“ $\Leftarrow$ ” Conversely, assume that some vector in D is a linear combination of the other vectors of D . We may assume that $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_k$. Then

$$\mathbf{v}_1 = s_2 \mathbf{v}_2 + s_3 \mathbf{v}_3 + \cdots + s_k \mathbf{v}_k$$

for some $s_2, s_3, \dots, s_k \in \mathbb{R}$, implying that

$$1\mathbf{v}_1 - s_2 \mathbf{v}_2 - s_3 \mathbf{v}_3 - \cdots - s_k \mathbf{v}_k = \mathbf{0}.$$

Thus there is a nontrivial linear combination of the vectors of D that vanishes, so D is dependent. ■

Proof. (continued)

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Thus there is a nontrivial linear combination of the vectors of D that vanishes, so D is dependent. ■

Suppose $U = \text{span}(S)$ for some set of vectors S . If S is dependent, then we can find a vector $\mathbf{v}$ in S that is a linear combination of the other vectors of S . Deleting $\mathbf{v}$ from S results in a set T with $\text{span}(T) = \text{span}(S) = U$.

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Proof.

Algorithm 6: Proof of Theorem

Input : 1. V : finite dimensional vector space spanned by m vectors
3. S : a spanning set of V

$S \rightarrow W$;

while W is dependent do

 Find out one $\mathbf{x} \in W$ that can be linearly represented by the rest;

$W \setminus \{\mathbf{x}\} \rightarrow W$;

 Dependent Lemma guarantees that the span of the new W remains to be V ;

end

Output: W , that is independent and spans V ; hence a basis of V .



Problem

Let

$$X_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad X_5 = \begin{bmatrix} 0 & 2 \\ 2 & -3 \end{bmatrix},$$

and let $U = \{X_1, X_2, X_3, X_4, X_5\}$. Then $\text{span}(U) = \mathbf{M}_{22}$. Find a basis of $\mathbf{M}_{22}$ from among the elements of U .

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and let $U = \{X_1, X_2, X_3, X_4, X_5\}$. Then $\text{span}(U) = \mathbf{M}_{22}$. Find a basis of $\mathbf{M}_{22}$ from among the elements of U .

Solution

Since U has five matrices and $\dim(\mathbf{M}_{22}) = 4$, U is dependent. Suppose

$$aX_1 + bX_2 + cX_3 + dX_4 + eX_5 = \mathbf{0}_{22}.$$

This gives us a homogeneous system of four equations in five variables, whose general solution is

$$a = -\frac{4}{3}t; \quad b = \frac{1}{3}t; \quad c = -\frac{2}{3}t; \quad d = 0; \quad e = t, \quad \text{for } t \in \mathbb{R}.$$

Solution (continued)

Taking $t = 3$ gives us

$$-4X_1 + X_2 - 2X_3 + 3X_5 = \mathbf{0}_{22}.$$

From this, we see that X_1 can be expressed as a linear combination of X_2 , X_3 and X_5 .

Let

$$B = \{X_2, X_3, X_4, X_5\}.$$

Then $\text{span}(B) = \text{span}(U) = \mathbf{M}_{22}$. If B is not independent, then apply the Dependent Lemma to find a subset of three matrices of B that spans $\mathbf{M}_{22}$. Since

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is an independent subset of $\mathbf{M}_{22}$, this contradicts the Fundamental Theorem. Therefore B is independent, and hence is a basis of $\mathbf{M}_{22}$. 

Theorem (Generalization of $\mathbb{R}^n$)

Let V be a finite dimensional vector space with $\dim(V) = n$, and suppose S is a subset of V containing n vectors. Then S is independent if and only if S spans V .

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Remark

This theorem can be used to simplify the arguments used in various problems covered.

Problem

Find a basis of $\mathcal{P}_2$ among the elements of the set

$$U = \{x^2 - 3x + 2, \quad 1 - 2x, \quad 2x^2 + 1, \quad 2x^2 - x - 3\}.$$

Solution

Since $|U| = 4 > 3 = \dim(\mathcal{P}_2)$, U is dependent.

Suppose $a(x^2 - 3x + 2) + b(1 - 2x) + c(2x^2 + 1) + d(2x^2 - x - 3) = 0$; then

$$(a + 2c + 2d)x^2 + (-3a - 2b - d)x + (2a + b + c - 3d) = 0.$$

This leads to a system of three equations in four variables that can be solved using gaussian elimination.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 0 \\ -3 & -2 & 0 & -1 & 0 \\ 2 & 1 & 1 & -3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

Thus $a = -2t$, $b = 3t$, $c = t$ and $d = 0$ for any $t \in \mathbb{R}$. Also, since each row of the reduced row-echelon matrix has a leading one, U spans $\mathcal{P}_2$.

Solution (continued)

Let $t = -1$. Then

$$2(x^2 - 3x + 2) - 3(1 - 2x) - (2x^2 + 1) = 0,$$

so any one of $\{x^2 - 3x + 2, 1 - 2x, 2x^2 + 1\}$ can be expressed as a linear combination of the other two. Let's remove $x^2 - 3x + 2$. Hence, set

$$B = \{1 - 2x, 2x^2 + 1, 2x^2 - x - 3\}.$$

Then $\text{span}(B) = \text{span}(U) = \mathcal{P}_2$. Since $|B| = 3 = \dim(\mathcal{P}_2)$, it follows from that B is independent. Therefore, $B \subseteq U$ is a basis of $\mathcal{P}_2$. ■

Problem

Let $V = \{A \in \mathbf{M}_{22} \mid A^T = A\}$. Then V is a vector space. Find a basis of V consisting of **invertible** matrices.

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Remark

Note that V is the set of 2×2 symmetric matrices, so

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

From this, we deduce that $\dim(V) = 3$. (Why?) Thus, a basis of V consisting of invertible matrices will consist of **three independent symmetric invertible matrices**.

Solution

There are many solutions. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix B is invertible, so one approach is to take linear combinations of A and C to produce two independent invertible matrices; for example

$$A + C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A - C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to verify that $S = \{A + C, A - C, B\}$ is an independent subset of 2×2 invertible symmetric matrices. Since $|S| = 3 = \dim(V)$, S spans V and is therefore a basis of V . 

Sums and Intersections

Definition

Let V be a vector space, and let U and W be subspaces of V . Then

1. $U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$ and is called the **sum** of U and W .

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Lemma

Prove that both $U + W$ and $U \cap W$ are subspaces of V .

Proof. (of $U + W$)

1. Since U and W are subspaces of V , $\mathbf{0}$, the zero vector of V , is an element of both U and W . Since $\mathbf{0} + \mathbf{0} = \mathbf{0}$, $\mathbf{0} \in U + W$.
2. Let $\mathbf{x}_1, \mathbf{x}_2 \in U + W$. Then $\mathbf{x}_1 = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{x}_2 = \mathbf{u}_2 + \mathbf{w}_2$ for some $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$. It follows that

$$\mathbf{x}_1 + \mathbf{x}_2 = (\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2).$$

Since U and W are subspaces of V , $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$, and therefore $\mathbf{x}_1 + \mathbf{x}_2 \in U + W$.

3. Let $\mathbf{x}_1 \in U + W$ and $k \in \mathbb{R}$. Then $\mathbf{x}_1 = \mathbf{u}_1 + \mathbf{w}_1$ for some $\mathbf{u}_1 \in U$ and $\mathbf{w}_1 \in W$. It follows that $k\mathbf{x}_1 = k(\mathbf{u}_1 + \mathbf{w}_1) = (k\mathbf{u}_1) + (k\mathbf{w}_1)$. Since U and W are subspaces of V , $k\mathbf{u}_1 \in U$ and $k\mathbf{w}_1 \in W$, and therefore $k\mathbf{x}_1 \in U + W$.

By the Subspace Test, $U + W$ is a subspace of V . ■

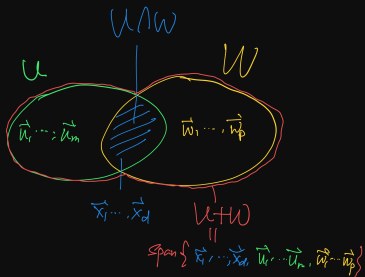
Theorem

If U and W are finite dimensional subspaces of a vector space V , then $U + W$ is finite dimensional and

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Remark

V need not be finite dimensional!



Proof.

$U \cap W$ is a subspace of the finite dimensional vector space U , so is finite dimensional, and has a finite basis $X = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d\}$. Since $X \subseteq U \cap W$, X can be extended to a finite basis B_U of U and a finite basis B_W of W :

$$B_U = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\} \quad \text{and} \quad B_W = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}.$$

Then

$$\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_d, \mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{w}_1, \dots, \mathbf{w}_p\} = U + W.$$

Proof. (continued)

What remains is to prove that

$$B = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_d, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$$

is a basis of $U + W$ since then it implies that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\Updownarrow$$

$$d + m + p = (d + m) + (d + p) - d$$

Proof. (continued)

To prove B is linearly independent, we need to show that

$$r_1 \mathbf{x}_1 + \cdots + r_d \mathbf{x}_d + s_1 \mathbf{u}_1 + \cdots + s_m \mathbf{u}_m + t_1 \mathbf{w}_1 + \cdots + t_p \mathbf{w}_p = \mathbf{0}.$$

which is equivalent to

$$\underbrace{r_1 \mathbf{x}_1 + \cdots + r_d \mathbf{x}_d + s_1 \mathbf{u}_1 + \cdots + s_m \mathbf{u}_m}_{\in U} = \underbrace{-t_1 \mathbf{w}_1 - \cdots - t_p \mathbf{w}_p}_{\in W}$$

Hence,

1. LHS $\in U \cap W$, which implies that $s_1 = \cdots = s_m = 0$.
2. RHS $\in U \cap W$, which implies that $t_1 = \cdots = t_p = 0$.

Finally,

$$r_1 \mathbf{x}_1 + \cdots + r_d \mathbf{x}_d = \mathbf{0}$$

implies that $r_1 = \cdots = r_d = 0$. This proves that B is independent. ■