Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations

§7-1. Examples and Elementary Properties

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Emory University, 2020 Fall

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

What is a Linear Transformation?

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Definition

Let V and W be vector spaces, and $T : V \to W$ a function. Then T is called a linear transformation if it satisfies the following two properties.

1. T preserves addition.
   For all $\vec{v}_1, \vec{v}_2 \in V$, $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$.
2. T preserves scalar multiplication.
   For all $\vec{v} \in V$ and $r \in \mathbb{R}$, $T(r\vec{v}) = rT(\vec{v})$.

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Remark

Note that the sum $\vec{v}_1 + \vec{v}_2$ is in V, while the sum $T(\vec{v}_1) + T(\vec{v}_2)$ is in W. Similarly, $r\vec{v}$ is scalar multiplication in V, while $rT(\vec{v})$ is scalar multiplication in W.

Theorem (Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$)

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is induced by an $m \times n$ matrix

$$A = \left[\begin{array}{cccc} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{array} \right],$$

where $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}^n$, and thus for each $\vec{x} \in \mathbb{R}^n$

$$T(\vec{x}) = A\vec{x}.$$

Example

$T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - z \end{bmatrix}$ for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$.

One can show that T preserves addition and scalar multiplication, and hence is a linear transformation. Therefore, the matrix that induces T is

$$A = \begin{bmatrix} T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Remark (Notation and Terminology)

1. If A is an $m \times n$ matrix, then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_A(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \in \mathbb{R}^n$$

   is the linear (or matrix) transformation induced by A.

2. Let V be a vector space. A linear transformation $T : V \to V$ is called a linear operator on V.

Example

Let V and W be vector spaces.

1. The zero transformation.
   $0 : V \to W$ is defined by $0(\vec{x}) = \vec{0}$ for all $\vec{x} \in V$.

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   $1_V : V \to V$ is defined by $1_V(\vec{x}) = \vec{x}$ for all $\vec{x} \in V$.

Examples and Problems

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3. The scalar operator on V. Let $a \in \mathbb{R}$.
   $s_a : V \to V$ is defined by $s_a(\vec{x}) = a\vec{x}$ for all $\vec{x} \in V$.

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   $s_a : V \to V$ is defined by $s_a(\vec{x}) = a\vec{x}$ for all $\vec{x} \in V$.

Problem

For vector spaces V and W, prove that the zero transformation, the identity operator, and the scalar operator are linear transformations.

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1. Let $\vec{u}, \vec{w} \in V$. Then $s_a(\vec{u}) = a\vec{u}$ and $s_a(\vec{w}) = a\vec{w}$. Now

$$s_a(\vec{u} + \vec{w}) = a(\vec{u} + \vec{w}) = a\vec{u} + a\vec{w} = s_a(\vec{u}) + s_a(\vec{w}),$$

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2. Let $\vec{u} \in V$ and $k \in \mathbb{R}$. Then $s_a(\vec{u}) = a\vec{u}$. Now

$$s_a(k\vec{u}) = ak\vec{u} = ka\vec{u} = ks_a(\vec{u}),$$

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   and thus s_a preserves scalar multiplication.

Since s_a preserves addition and scalar multiplication, s_a is a linear transformation. ■

Example (Matrix transposition)

Let $R : \mathbf{M}_{nn} \to \mathbf{M}_{nn}$ be a transformation defined by

$$R(A) = A^T \text{ for all } A \in \mathbf{M}_{nn}.$$

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2. Let $A \in \mathbf{M}_{nn}$ and let $k \in \mathbb{R}$. Then $R(A) = A^T$, and

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Since R preserves addition and scalar multiplication, R is a linear transformation.

Example (Evaluation at a)

For each $a \in \mathbb{R}$, the transformation $E_a : \mathcal{P}_n \to \mathbb{R}$ is defined by

$$E_a(p) = p(a) \text{ for all } p \in \mathcal{P}_n.$$

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1. Let $p, q \in \mathcal{P}_n$. Then $E_a(p) = p(a)$ and $E_a(q) = q(a)$, so

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2. Let $p \in \mathcal{P}_n$ and $k \in \mathbb{R}$. Then $E_a(p) = p(a)$ and

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Since E_a preserves addition and scalar multiplication, E_a is a linear transformation.

Problem

Let $\mathrm{S} : \mathbf{M}_{\mathrm{nn}} \to \mathbb{R}$ be a transformation defined by

$$\mathrm{S}(\mathrm{A}) = \mathrm{tr}(\mathrm{A}) \text{ for all } \mathrm{A} \in \mathbf{M}_{\mathrm{nn}}.$$

Prove that S is a linear transformation.

Solution

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

$$S(A) = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad S(B) = \sum_{i=1}^{n} b_{ii}.$$

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$$S(A) = \sum_{i=1}^{n} a_{ii} \quad \text{and} \quad S(B) = \sum_{i=1}^{n} b_{ii}.$$

1. Since $A + B = [a_{ij} + b_{ij}]$,

$$S(A+B) = \mathrm{tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \left(\sum_{i=1}^{n} a_{ii}\right)+\left(\sum_{i=1}^{n} b_{ii}\right) = S(A)+S(B).$$

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Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

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2. Let $k \in \mathbb{R}$. Since $kA = [ka_{ij}]$,

$$S(kA) = \mathrm{tr}(kA) = \sum_{i=1}^{n} ka_{ii} = k\sum_{i=1}^{n} a_{ii} = kS(A).$$

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Therefore, S preserves addition and scalar multiplication, and thus is a linear transformation. ■

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Theorem

Let V and W be vector spaces, and $T : V \to W$ a linear transformation. Then

1. T preserves the zero vector. $T(\vec{0}) = \vec{0}$.
2. T preserves additive inverses. For all $\vec{v} \in V$, $T(-\vec{v}) = -T(\vec{v})$.
3. T preserves linear combinations. For all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in V$ and all $k_1, k_2, \ldots, k_m \in \mathbb{R}$,

$$T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_m\vec{v}_m) = k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_mT(\vec{v}_m).$$

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Proof.

1. Let $\vec{0}_V$ denote the zero vector of V and let $\vec{0}_W$ denote the zero vector of W. We want to prove that $T(\vec{0}_V) = \vec{0}_W$. Let $\vec{x} \in V$. Then $0\vec{x} = \vec{0}_V$ and

$$T(\vec{0}_V) = T(0\vec{x}) = 0T(\vec{x}) = \vec{0}_W.$$

Proof. (continued)

2. Let $\vec{v} \in V$; then $-\vec{v} \in V$ is the additive inverse of $\vec{v}$, so $\vec{v} + (-\vec{v}) = \vec{0}_V$. Thus

$$\begin{aligned} T(\vec{v} + (-\vec{v})) &= T(\vec{0}_V) \\ T(\vec{v}) + T(-\vec{v})) &= \vec{0}_W \\ T(-\vec{v}) &= \vec{0}_W - T(\vec{v}) = -T(\vec{v}). \end{aligned}$$

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One of the keys to doing problems involving linear transformations is to make effective use of the fact that linear transformations preserve linear combinations.

Problem

Let $T : \mathcal{P}_2 \to \mathbb{R}$ be a linear transformation such that

$$T(x^2 + x) = -1; T(x^2 - x) = 1; T(x^2 + 1) = 3.$$

Find $T(4x^2 + 5x - 3)$.

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Solution (first)

Suppose $a(x^2 + x) + b(x^2 - x) + c(x^2 + 1) = 4x^2 + 5x - 3$. Then

$$(a + b + c)x^2 + (a - b)x + c = 4x^2 + 5x - 3.$$

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Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$.

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Solving for a, b, and c results in the unique solution $a = 6$, $b = 1$, $c = -3$. Thus

$$\begin{aligned} T(4x^2 + 5x - 3) &= T\left(6(x^2 + x) + (x^2 - x) - 3(x^2 + 1)\right) \\ &= 6T(x^2 + x) + T(x^2 - x) - 3T(x^2 + 1) \\ &= 6(-1) + 1 - 3(3) = -14. \end{aligned}$$

$\blacksquare$

Solution (second)

Notice that $S = \{x^2 + x, x^2 - x, x^2 + 1\}$ is a basis of $\mathcal{P}_2$, and thus x^2, x, and 1 can each be written as a linear combination of elements of S.

Solution (second)

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$$\begin{aligned} x^2 &= \tfrac{1}{2}(x^2 + x) + \tfrac{1}{2}(x^2 - x) \\ x &= \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x) \\ 1 &= (x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x). \end{aligned}$$

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&= \tfrac{1}{2}(-1) + \tfrac{1}{2}(1) = 0. \\
T(x) &= T\left(\tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) = \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\
&= \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = -1. \\
T(1) &= T\left((x^2 + 1) - \tfrac{1}{2}(x^2 + x) - \tfrac{1}{2}(x^2 - x)\right) \\
&= T(x^2 + 1) - \tfrac{1}{2}T(x^2 + x) - \tfrac{1}{2}T(x^2 - x) \\
&= 3 - \tfrac{1}{2}(-1) - \tfrac{1}{2}(1) = 3.
\end{aligned}$$

$$\Downarrow$$

$$T(4x^2 + 5x - 3) = 4T(x^2) + 5T(x) - 3T(1) = 4(0) + 5(-1) - 3(3) = -14.$$

$\blacksquare$

Remark

The advantage of this solution over Solution 1 is that if you were now asked to find $T(-6x^2 - 13x + 9)$, it is easy to use $T(x^2) = 0$, $T(x) = -1$ and $T(1) = 3$:

$$\begin{aligned} T(-6x^2 - 13x + 9) &= -6T(x^2) - 13T(x) + 9T(1) \\ &= -6(0) - 13(-1) + 9(3) = 13 + 27 = 40. \end{aligned}$$

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More generally,

$$\begin{aligned} T(ax^2 + bx + c) &= aT(x^2) + bT(x) + cT(1) \\ &= a(0) + b(-1) + c(3) = -b + 3c. \end{aligned}$$

Definition (Equality of linear transformations)

Let V and W be vector spaces, and let S and T be linear transformations from V to W. Then $S = T$ if and only if, for every $\vec{v} \in V$,

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Theorem

Let V and W be vector spaces, where

$$V = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}.$$

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Remark

This theorem tells us that a linear transformation is completely determined by its actions on a spanning set.

Proof.

We must show that $S(\vec{v}) = T(\vec{v})$ for each $\vec{v} \in V$. Let $\vec{v} \in V$. Then (since V is spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$), there exist $k_1, k_2, \ldots, k_n \in \mathbb{R}$ so that

$$\vec{v} = k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n.$$

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It follows that

$$\begin{aligned} S(\vec{v}) &= S(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\ &= k_1S(\vec{v}_1) + k_2S(\vec{v}_2) + \cdots + k_nS(\vec{v}_n) \\ &= k_1T(\vec{v}_1) + k_2T(\vec{v}_2) + \cdots + k_nT(\vec{v}_n) \\ &= T(k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n) \\ &= T(\vec{v}). \end{aligned}$$

Therefore, $S = T$. ■

Constructing Linear Transformations

Constructing Linear Transformations

Theorem

Let V and W be vector spaces, let $B = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ be a basis of V, and let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ be (not necessarily distinct) vectors of W. Then there exists a unique linear transformation $T : V \to W$ such that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$. Furthermore, if

$$\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$$

is a vector of V, then

$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n.$$

Proof.

Suppose $\vec{v} \in V$. Since B is a basis, there exist unique scalars $k_1, k_2, \ldots, k_n \in \mathbb{R}$ so that $\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$. We now define $T : V \to W$ by

$$T(\vec{v}) = k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n$$

for each $\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n$ in V. From this definition, $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$.

To prove that T is a linear transformation, prove that T preserves addition and scalar multiplication. Let $\vec{v}, \vec{u} \in V$. Then

$$\vec{v} = k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n \quad \text{and} \quad \vec{u} = \ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n$$

for some $k_1, k_2, \ldots, k_n \in \mathbb{R}$ and $\ell_1, \ell_2, \ldots, \ell_n \in \mathbb{R}$.

Proof. (continued)

$$
\begin{aligned}
T(v \vec{+} u) &= T[(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + (\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n)] \\
&= T[(k_1 + \ell_1)\vec{b}_1 + (k_2 + \ell_2)\vec{b}_2 + \cdots + (k_n + \ell_n)\vec{b}_n] \\
&= (k_1 + \ell_1)\vec{w}_1 + (k_2 + \ell_2)\vec{w}_2 + \cdots + (k_n + \ell_n)\vec{w}_n \\
&= (k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) + (\ell_1\vec{w}_1 + \ell_2\vec{w}_2 + \cdots + \ell_n\vec{w}_n) \\
&= T(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + T(\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n) \\
&= T(\vec{v}) + T(\vec{u}).
\end{aligned}
$$

Therefore, T preserves addition.

Proof. (continued)

$$\begin{aligned}
T(v \vec{+} u) &= T[(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + (\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n)] \\
&= T[(k_1 + \ell_1)\vec{b}_1 + (k_2 + \ell_2)\vec{b}_2 + \cdots + (k_n + \ell_n)\vec{b}_n] \\
&= (k_1 + \ell_1)\vec{w}_1 + (k_2 + \ell_2)\vec{w}_2 + \cdots + (k_n + \ell_n)\vec{w}_n \\
&= (k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) + (\ell_1\vec{w}_1 + \ell_2\vec{w}_2 + \cdots + \ell_n\vec{w}_n) \\
&= T(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) + T(\ell_1\vec{b}_1 + \ell_2\vec{b}_2 + \cdots + \ell_n\vec{b}_n) \\
&= T(\vec{v}) + T(\vec{u}).
\end{aligned}$$

Therefore, T preserves addition. Let $\vec{v}$ be as already defined and let $r \in \mathbb{R}$. Then

$$\begin{aligned}
T(r\vec{v}) &= T[r(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n)] \\
&= T[(rk_1)\vec{b}_1 + (rk_2)\vec{b}_2 + \cdots + (rk_n)\vec{b}_n] \\
&= (rk_1)\vec{w}_1 + (rk_2)\vec{w}_2 + \cdots + (rk_n)\vec{w}_n \\
&= r(k_1\vec{w}_1 + k_2\vec{w}_2 + \cdots + k_n\vec{w}_n) \\
&= rT(k_1\vec{b}_1 + k_2\vec{b}_2 + \cdots + k_n\vec{b}_n) \\
&= rT(\vec{v}).
\end{aligned}$$

Therefore, T preserves scalar multiplication.

Proof. (continued)

Finally, the previous Theorem guarantees that T is unique: since B is a basis (and hence a spanning set), the action of T is completely determined by the fact that $T(\vec{b}_i) = \vec{w}_i$ for each i, $1 \leq i \leq n$. This completes the proof of the theorem. $\blacksquare$

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Remark

The significance of this Theorem is that it gives us the ability to define linear transformations between vector spaces, a useful tool in what follows.

Problem

$B = \{1+x, x+x^2, 1+x^2\}$ is a basis of $\mathcal{P}_2$ (you should be able to prove this). Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(elements of $\mathbf{M}_{22}$). Find a linear transformation $T : \mathcal{P}_2 \to \mathbf{M}_{22}$ so the

$$T(1+x) = A_1, T(x+x^2) = A_2, \quad \text{and} \quad T(1+x^2) = A_3,$$

i.e., for $a + bx + cx^2 \in \mathcal{P}_2$, find $T(a+bx+cx^2)$.

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i.e., for $a + bx + cx^2 \in \mathcal{P}_2$, find $T(a+bx+cx^2)$.

Solution

Notice that $(1+x)+(x+x^2)-(1+x^2)=2x$, and thus

$$x = \tfrac{1}{2}(1+x)+\tfrac{1}{2}(x+x^2)-\tfrac{1}{2}(1+x^2),$$

$$\Downarrow$$

$$\begin{aligned} T(x) &= \tfrac{1}{2}T(1+x)+\tfrac{1}{2}T(x+x^2)-\tfrac{1}{2}T(1+x^2) \\ &= \tfrac{1}{2}A_1+\tfrac{1}{2}A_2-\tfrac{1}{2}A_3 \\ &= \tfrac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}+\tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}-\tfrac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}=\tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

Solution (continued)

Next, $1 = (1+x) - x$, so $T(1) = T(1+x) - T(x)$, and thus

$$T(1) = A_1 - \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

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Finally, $x^2 = (x + x^2) - x$, so $T(x^2) = T(x + x^2) - T(x)$, and thus

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Therefore,

$$\begin{aligned} T(a+bx+cx^2) &= aT(1) + bT(x) + cT(x^2) \\ &= \frac{a}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{b}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{c}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} a+b-c & -a+b+c \\ -a+b+c & a-b+c \end{bmatrix}. \end{aligned}$$

■

Problem

Let V be a vector space, T a linear operator on V, and $\mathbf{v}, \mathbf{w} \in V$. Suppose that

$$T(\mathbf{v} + \mathbf{w}) = \mathbf{v} - 2\mathbf{w} \quad \text{and} \quad T(2\mathbf{v} - \mathbf{w}) = 2\mathbf{v}.$$

Find $T(\mathbf{v})$ and $T(\mathbf{w})$.

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Find $T(\mathbf{v})$ and $T(\mathbf{w})$.

Solution (final answer)

$T(\mathbf{v}) = \mathbf{v} - \frac{2}{3}\mathbf{w}$ and $T(\mathbf{w}) = -\frac{4}{3}\mathbf{w}$. ■