Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-2. Kernel and Image

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

What are the Kernel and the Image?

Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation.

1. The kernel of T (sometimes called the null space of T) is defined to be the set
$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
2. The image of T is defined to be the set
$$\operatorname{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

Remark

If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A, then

- $\ker(T_A) = \operatorname{null}(A)$;
- $\operatorname{im}(T_A) = \operatorname{im}(A)$.

Example

Let $T : \mathcal{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1.$$

$$\begin{aligned}
\ker(T) &= \{p(x) \in \mathcal{P}_1 \mid p(1) = 0\} \\
&= \{ax + b \mid a, b \in \mathbb{R} \quad \text{and} \quad a + b = 0\} \\
&= \{ax - a \mid a \in \mathbb{R}\}.
\end{aligned}$$

$$\begin{aligned}
\operatorname{im}(T) &= \{p(1) \mid p(x) \in \mathcal{P}_1\} \\
&= \{a + b \mid ax + b \in \mathcal{P}_1\} \\
&= \{a + b \mid a, b \in \mathbb{R}\} \\
&= \mathbb{R}.
\end{aligned}$$

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then $\ker(T)$ is a subspace of V and $\mathrm{im}(T)$ is a subspace of W.

Proof. (that $\ker(T)$ is a subspace of V)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Since $T(\vec{0}_V) = \vec{0}_W$, $\vec{0}_V \in \ker(T)$.
2. Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$. Then $T(\vec{v}_1) = \vec{0}$, $T(\vec{v}_2) = \vec{0}$, and
$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}.$$
Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.
3. Let $\vec{v}_1 \in \ker(T)$ and let $k \in \mathbb{R}$. Then $T(\vec{v}_1) = \vec{0}$, and
$$T(k\vec{v}_1) = kT(\vec{v}_1) = k(\vec{0}) = \vec{0}.$$
Thus $k\vec{v}_1 \in \ker(T)$.

By the Subspace Test, $\ker(T)$ is a subspace of V. ∎

Proof. (that $\mathrm{im}(T)$ is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Since $T(\vec{0}_V) = \vec{0}_W$, $\vec{0}_W \in \mathrm{im}(T)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \mathrm{im}(T)$. Then there exist $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, and thus
$$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2).$$
Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{im}(T)$.
3. Let $\vec{w}_1 \in \mathrm{im}(V)$ and let $k \in \mathbb{R}$. Then there exists $\vec{v}_1 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, and
$$k\vec{w}_1 = kT(\vec{v}_1) = T(k\vec{v}_1).$$
Since $k\vec{v}_1 \in V$, $k\vec{w}_1 \in \mathrm{im}(T)$.

By the Subspace Test, $\mathrm{im}(T)$ is a subspace of W. ■

Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. The dimension of $\ker(T)$, $\dim(\ker(T))$ is called the nullity of T and is denoted $\text{nullity}(T)$, i.e.,

$$\text{nullity}(T) = \dim(\ker(T)).$$

2. The dimension of $\text{im}(T)$, $\dim(\text{im}(T))$ is called the rank of T and is denoted rank (T), i.e.,

$$\text{rank}(T) = \dim(\text{im}(T)).$$

Example

If A is an $m \times n$ matrix, then

$$\mathrm{im}(T_A) = \mathrm{im}(A) = \mathrm{col}(A).$$

It follows that

$$\mathrm{rank}\,(T_A) = \dim(\mathrm{im}(T_A)) = \dim(\mathrm{col}(A)) = \mathrm{rank}\,(A).$$

Also, $\ker(T_A) = \mathrm{null}(A)$, so

$$\mathrm{nullity}(T_A) = \dim(\mathrm{null}(A)) = n - \mathrm{rank}\,(A).$$

Finding bases of the kernel and the image

Example (continued)

For the linear transformation T defined by $T : \mathcal{P}_1 \to \mathbb{R}$

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1,$$

we found that

$$\begin{aligned} \ker(T) &= \{ax - a \mid a \in \mathbb{R}\} \quad \text{and} \\ \operatorname{im}(T) &= \mathbb{R}. \end{aligned}$$

From this, we see that $\ker(T) = \operatorname{span}\{(x-1)\}$; since $\{(x-1)\}$ is an independent subset of $\mathcal{P}_1$, $\{(x-1)\}$ is a basis of $\ker(T)$. Thus

$$\dim(\ker(T)) = 1 = \operatorname{nullity}(T).$$

Since $\operatorname{im}(T) = \mathbb{R}$,

$$\dim(\operatorname{im}(T)) = 1 = \operatorname{rank}(T).$$

Problem

Let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Then T is a linear transformation (you should be able to prove this). Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

Solution

Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$. Then

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us a system of four equations in the four variables a, b, c, d:

$$a + b = 0; b + c = 0; c + d = 0; d + a = 0.$$

Solution (continued)

This system has solution $a = -t, b = t, c = -t, d = t$ for any $t \in \mathbb{R}$, and thus

$$\ker(T) = \left\{ \begin{bmatrix} -t & t \\ -t & t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

Let $B_k = \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$. Since B_k is an independent subset of $\mathbf{M}_{22}$ and $\operatorname{span}(B) = \ker(T)$, B_k is a basis of $\ker(T)$.

$$\begin{aligned} \operatorname{im}(T) &= \left\{ \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\} \\ &= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Solution (continued)

S is a dependent subset of $\mathbf{M}_{22}$, but (check this yourselves)

$$B_i = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is an independent subset of S. Since $\text{span}(B_i) = \text{span}(S) = \text{im}(T)$ and B_i is independent, B_i is a basis of $\text{im}(T)$. ∎

Surjections and Injections

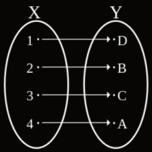
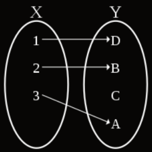
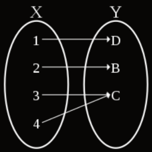
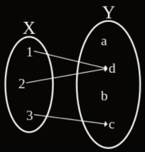
Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. T is onto (or surjective) if $\mathrm{im}(T) = W$.
2. T is one-to-one (or injective) if, for $\vec{v}, \vec{w} \in V$, $T(\vec{v}) = T(\vec{w})$ implies that $\vec{v} = \vec{w}$.

Example

Let V be a vector space. Then the identity operator on V, $1_V : V \to V$, is one-to-one and onto.

	surjective	non-surjective
injective	 bijective	 injective-only
non-injective	 surjective-only	 general

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

Proof.

($\Rightarrow$) Let $\vec{v} \in \ker(T)$. Then

$$T(\vec{v}) = \vec{0} = T(\vec{0}).$$

Since is one-to-one, $\vec{v} = \vec{0}$. But $\vec{v}$ is an arbitrary element of $\ker(T)$, and thus $\ker T = \{\vec{0}\}$.

($\Leftarrow$) Conversely, suppose that $\ker(T) = \{\vec{0}\}$, and let $\vec{v}, \vec{w} \in V$ be such that

$$T(\vec{v}) = T(\vec{w}).$$

Then $T(\vec{v}) - T(\vec{w}) = \vec{0}$, and since T is a linear transformation

$$T(\vec{v} - \vec{w}) = \vec{0}.$$

By definition, $\vec{v} - \vec{w} \in \ker(T)$, implying that $\vec{v} - \vec{w} = \vec{0}$. Therefore $\vec{v} = \vec{w}$, and hence T is one-to-one. ∎

Problem

Let $T : \mathbf{M}_{22} \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Prove that T is onto but not one-to-one.

Proof.

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Since $T\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, T is onto.

Observe that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \ker(T)$, so $\ker(T) \neq \vec{0}_{22}$. By the previous Theorem, T is not one-to-one. ■

Problem

Suppose U is an invertible $\mathrm{m} \times \mathrm{m}$ matrix and let $\mathrm{T} : \mathbf{M}_{\mathrm{mn}} \to \mathbf{M}_{\mathrm{mn}}$ be defined by

$$\mathrm{T}(\mathrm{A}) = \mathrm{UA} \text{ for all } \mathrm{A} \in \mathbf{M}_{\mathrm{mn}}.$$

Then T is a linear transformation (this is left to you to verify). Prove that T is one-to-one and onto.

Proof.

Suppose $\mathrm{A}, \mathrm{B} \in \mathbf{M}_{\mathrm{mn}}$ and that $\mathrm{T}(\mathrm{A}) = \mathrm{T}(\mathrm{B})$. Then $\mathrm{UA} = \mathrm{UB}$; since U is invertible

$$\begin{aligned} \mathrm{U}^{-1}(\mathrm{UA}) &= \mathrm{U}^{-1}(\mathrm{UB}) \\ (\mathrm{U}^{-1}\mathrm{U})\mathrm{A} &= (\mathrm{U}^{-1}\mathrm{U})\mathrm{B} \\ \mathrm{I}_{\mathrm{mm}}\mathrm{A} &= \mathrm{I}_{\mathrm{mm}}\mathrm{B} \\ \mathrm{A} &= \mathrm{B}. \end{aligned}$$

Therefore, T is one-to-one.

Proof. (continued)

To prove that T is onto, let $B \in \mathbf{M}_{mn}$ and let $A = U^{-1}B$. Then

$$T(A) = UA = U(U^{-1}B) = (UU^{-1})B = I_{mm}B = B,$$

and therefore T is onto. ∎

Problem

Let $S : \mathcal{P}_2 \to \mathbf{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2 + bx + c \in \mathcal{P}_2.$$

Prove that S is one-to-one but not onto.

Proof.

By definition,

$$\ker(S) = \{ax^2 + bx + c \in \mathcal{P}_2 \mid a+b=0, a+c=0, b-c=0, b+c=0\}.$$

Suppose $p(x) = ax^2 + bx + c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unique solution is $a = b = c = 0$, $\ker(S) = \{\vec{0}\}$, and thus S is one-to-one.

Proof. (continued)

To show that S is not onto, show that $\mathrm{im}(S) \neq \mathcal{P}_2$; i.e., find a matrix $A \in \mathbf{M}_{22}$ such that for every $p(x) \in \mathcal{P}_2$, $S(p(x)) \neq A$. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix},$$

and suppose $p(x) = ax^2 + bx + c \in \mathcal{P}_2$ is such that $S(p(x)) = A$. Then

$$\begin{array}{ll} a + b = 0 & a + c = 1 \\ b - c = 0 & b + c = 2 \end{array}$$

Solving this system

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

Since the system is inconsistent, there is no $p(x) \in \mathcal{P}_2$ so that $S(p(x)) = A$, and therefore S is not onto. ∎

Problem (One-to-one linear transformations preserve independent sets)

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is one-to-one and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an independent subset of V, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is an independent subset of W.

Proof.

Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Suppose that

$$a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \cdots + a_k T(\vec{v}_k) = \vec{0}_W$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Since linear transformations preserve linear combinations (addition and scalar multiplication),

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W.$$

Now, since T is one-to-one, $\ker(T) = \{\vec{0}_V\}$, and thus

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V.$$

However, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is independent, and hence $a_1 = a_2 = \cdots = a_k = 0$. Therefore, $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent. ∎

Problem (Onto linear transformations preserve spanning sets)

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is onto and $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, then

$$W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Proof.

Suppose that T is onto and let $\mathbf{w} \in W$. Then there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Since $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, there exist $a_1, a_2, \ldots a_k \in \mathbb{R}$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$. Using the fact that T is a linear transformation,

$$\begin{aligned} \mathbf{w} = T(\mathbf{v}) &= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_kT(\mathbf{v}_k), \end{aligned}$$

i.e., $\mathbf{w} \in \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$, and thus

$$W \subseteq \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Since $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k) \in W$, it follows that $\text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq W$, and therefore $W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$. ■

Suppose A is an $m \times n$ matrix. How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is onto? How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one?

Theorem

Let A be an $m \times n$ matrix, and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear transformation induced by A.

1. T_A is onto if and only if $\text{rank}\,(A) = m$.
2. T_A is one-to-one if and only if $\text{rank}\,(A) = n$.

Proof. (sketch)

1. T_A is onto if and only if $\text{im}(T_A) = \mathbb{R}^m$. This is equivalent to $\text{col}(A) = \mathbb{R}^m$, which occurs if and only if $\dim(\text{col}(A)) = m$, i.e., $\text{rank}\,(A) = m$.
2. $\ker(T_A) = \text{null}(A)$, and $\text{null}(A) = \{\vec{0}\}$ if and only if $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. Thus and row echelon form of A has a leading one in every column, which occurs if and only if $\text{rank}\,(A) = n$. ■

The Dimension Theorem (Rank-Nullity Theorem)

Suppose A is an $m \times n$ matrix with rank r. Since $\mathrm{im}(T_A) = \mathrm{col}(A)$,

$$\dim(\mathrm{im}(T_A)) = \mathrm{rank}\,(A) = r.$$

We also know that $\ker(T_A) = \mathrm{null}(A)$, and that $\dim(\mathrm{null}(A)) = n - r$. Thus,

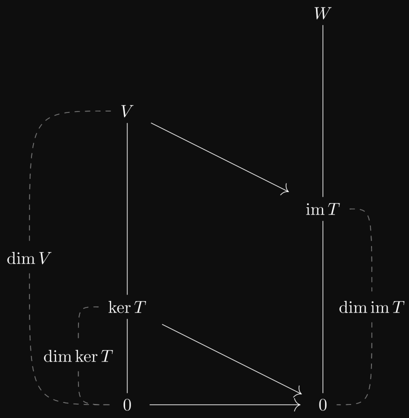
$$\dim(\mathrm{im}(T_A)) + \dim(\ker(T_A)) = n = \dim\ \mathbb{R}^n.$$

Theorem (Dimension Theorem (Rank-Nullity Theorem))

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If $\ker(T)$ and $\mathrm{im}(T)$ are both finite dimensional, then V is finite dimensional, and

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)).$$

Equivalently, $\dim(V) = \mathrm{nullity}(T) + \mathrm{rank}\,(T)$.


W
V
$\operatorname{im} T$
$\dim V$
$\ker T$
$\dim \operatorname{im} T$
$\dim \ker T$
0
0

Proof. (Outline)

Let $\vec{w} \in \mathrm{im}(T)$; then $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$. Suppose

$$\left\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_r)\right\}$$

is a basis of $\mathrm{im}(T)$, and that

$$\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}$$

is a basis of $\ker(T)$. We define

$$B = \left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_r, \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}.$$

To prove that B is a basis of V, it remains to prove that B spans V and that B is linearly independent.

Since B is independent and spans V, B is a basis of V, implying V is finite dimensional (V is spanned by a finite set of vectors). Furthermore, $|B| = r + k$, so

$$\dim(V) = \dim(\mathrm{im}(T)) + \dim(\ker(T)).$$

$\blacksquare$

Remark

1. It is not an assumption of the theorem that V is finite dimensional. Rather, it is a consequence of the assumption that both im(T) and ker(T) are finite dimensional.
2. As a consequence of the Dimension Theorem, if V is a finite dimensional vector space and either dim(ker(T)) or dim(im(T)) is known, then the other can be easily found.

Example

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If V is finite dimensional, then it follows that

$$\dim(\ker(T)) \leq \dim(V) \quad \text{and} \quad \dim(\operatorname{im}(T)) \leq \dim(V).$$

Problem

For $a \in \mathbb{R}$, recall that the linear transformation $E_a : \mathcal{P}_n \to \mathbb{R}$, the evaluation map at a, is defined as

$$E_a(p(x)) = p(a) \text{ for all } p(x) \in \mathcal{P}_n.$$

Prove that E_a is onto, and that

$$B = \{(x-a), (x-a)^2, (x-a)^3, \dots, (x-a)^n\}$$

is a basis of $\ker(E_a)$.

Proof.

Let $t \in \mathbb{R}$, and choose $p(x) = t \in \mathcal{P}_n$. Then $p(a) = t$, so $E_a(p(x)) = t$, i.e., E_a is onto.

By the Dimension Theorem,

$$n + 1 = \dim(\mathcal{P}_n) = \dim(\ker(E_a)) + \dim(\operatorname{im}(E_a)).$$

Since E_a is onto, $\dim(\operatorname{im}(E_a)) = \dim(\mathbb{R}) = 1$, and thus $\dim(\ker(E_a)) = n$.

It now suffices to find n independent polynomials in $\ker(E_a)$.

Note that $(x - a)^j \in \ker(E_a)$ for $j = 1, 2, \ldots, n$, so $B \subseteq \ker(E_a)$.

Furthermore, B is independent because the polynomials in B have distinct degrees.

Since $|B| = n = \dim(\ker(E_a))$, B spans $\ker(E_a)$.

Therefore, B is a basis of $\ker(E_a)$. $\blacksquare$

Theorem

Let V and W be vector spaces, $T : V \to W$ a linear transformation, and

$$B = \left\{ \vec{b}_1, \vec{b}_2, \ldots, \vec{b}_r, \vec{b}_{r+1}, \vec{b}_{r+2}, \ldots, \vec{b}_n \right\}$$

a basis of V with the property that $\left\{ \vec{b}_{r+1}, \vec{b}_{r+2}, \ldots, \vec{b}_n \right\}$ is a basis of $\ker(T)$. Then

$$\left\{ T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_r) \right\}$$

is a basis of $\operatorname{im}(T)$, and therefore $r = \operatorname{rank}(T)$.

Remark (How is this useful?)

Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. If you find a basis of $\ker(T)$, then this may be used to find a basis of $\operatorname{im}(T)$: extend the basis of $\ker(T)$ to a basis of V; applying the transformation T to each of the vectors that was added to the basis of $\ker(T)$ produces a set of vectors that is a basis of $\operatorname{im}(T)$.

Problem

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be a linear transformation defined by

$$T(X) = XA - AX \text{ for all } X \in \mathbf{M}_{22}.$$

Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

Solution

First note that by the Dimension Theorem,

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\mathbf{M}_{22}) = 4.$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned} T(X) &= AX - XA \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} \end{aligned}$$

Solution (continued)

If $X \in \ker(T)$, then $T(X) = \vec{0}_{22}$ so

$$c - b = 0, d - a = 0, a - d = 0, b - c = 0.$$

This system of four equations in four variables has general solution $a = s, b = t, c = t, d = s$ for $s, t \in \mathbb{R}$. Therefore,

$$\ker(T) = \left\{ \left[\begin{array}{cc} s & t \\ t & s \end{array} \right] \;\middle|\; s, t \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}.$$

Let $B_k = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$. Since B_k is independent and spans $\ker(T)$, B_k is a basis of $\ker(T)$.

To find a basis of $\operatorname{im}(T)$, extend the basis of $\ker(T)$ to a basis of $\mathbf{M}_{22}$: here is one such basis

$$\left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \right\}.$$

Solution (continued)

Thus

$$B_i = \left\{ T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

is a basis of $\mathrm{im}(T)$. ■