Math 221: LINEAR ALGEBRA

Chapter 7. Linear Transformations
§7-2. Kernel and Image

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

What are the Kernel and the Image?

Finding Bases of the Kernel and the Image

Surjections and Injections

The Dimension Theorem (Rank-Nullity Theorem)

What are the Kernel and the Image?

What are the Kernel and the Image?

Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation.

1. The kernel of T (sometimes called the null space of T) is defined to be the set
$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
2. The image of T is defined to be the set
$$\mathrm{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

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Remark

If A is an $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is the linear transformation induced by A, then

- $\ker(T_A) = \operatorname{null}(A)$;
- $\operatorname{im}(T_A) = \operatorname{im}(A)$.

Example

Let $T : \mathcal{P}_1 \to \mathbb{R}$ be the linear transformation defined by

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1.$$

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$$\begin{aligned} \ker(T) &= \{p(x) \in \mathcal{P}_1 \mid p(1) = 0\} \\ &= \{ax + b \mid a, b \in \mathbb{R} \quad \text{and} \quad a + b = 0\} \\ &= \{ax - a \mid a \in \mathbb{R}\}. \end{aligned}$$

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$$\begin{aligned} \text{im}(T) &= \{p(1) \mid p(x) \in \mathcal{P}_1\} \\ &= \{a + b \mid ax + b \in \mathcal{P}_1\} \\ &= \{a + b \mid a, b \in \mathbb{R}\} \\ &= \mathbb{R}. \end{aligned}$$

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then $\ker(T)$ is a subspace of V and $\text{im}(T)$ is a subspace of W.

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Proof. (that ker(T) is a subspace of V)

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2. Let $\vec{v}_1, \vec{v}_2 \in \ker(T)$. Then $T(\vec{v}_1) = \vec{0}$, $T(\vec{v}_2) = \vec{0}$, and

$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}.$$

   Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.

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   Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.
3. Let $\vec{v}_1 \in \ker(T)$ and let $k \in \mathbb{R}$. Then $T(\vec{v}_1) = \vec{0}$, and

   $$T(k\vec{v}_1) = kT(\vec{v}_1) = k(\vec{0}) = \vec{0}.$$

   Thus $k\vec{v}_1 \in \ker(T)$.

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Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then $\ker(T)$ is a subspace of V and $\mathrm{im}(T)$ is a subspace of W.

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$$T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}.$$
Thus $\vec{v}_1 + \vec{v}_2 \in \ker(T)$.
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$$T(k\vec{v}_1) = kT(\vec{v}_1) = k(\vec{0}) = \vec{0}.$$
Thus $k\vec{v}_1 \in \ker(T)$.

By the Subspace Test, $\ker(T)$ is a subspace of V. ■

Proof. (that im(T) is a subspace of W)

1. Let $\vec{0}_{\mathrm{V}}$ and $\vec{0}_{\mathrm{W}}$ denote the zero vectors of V and W, respectively. Since $\mathrm{T}(\vec{0}_{\mathrm{V}}) = \vec{0}_{\mathrm{W}}$, $\vec{0}_{\mathrm{W}} \in \mathrm{im}(\mathrm{T})$.

Proof. (that $\mathrm{im}(T)$ is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Since $T(\vec{0}_V) = \vec{0}_W$, $\vec{0}_W \in \mathrm{im}(T)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \mathrm{im}(T)$. Then there exist $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, and thus

$$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2).$$

   Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{im}(T)$.

Proof. (that im(T) is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Since $T(\vec{0}_V) = \vec{0}_W$, $\vec{0}_W \in \text{im}(T)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \text{im}(T)$. Then there exist $\vec{v}_1, \vec{v}_2 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, $T(\vec{v}_2) = \vec{w}_2$, and thus

   $$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2).$$

   Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \text{im}(T)$.
3. Let $\vec{w}_1 \in \text{im}(V)$ and let $k \in \mathbb{R}$. Then there exists $\vec{v}_1 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, and

   $$k\vec{w}_1 = kT(\vec{v}_1) = T(k\vec{v}_1).$$

   Since $k\vec{v}_1 \in V$, $k\vec{w}_1 \in \text{im}(T)$.

Proof. (that $\mathrm{im}(T)$ is a subspace of W)

1. Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Since $T(\vec{0}_V) = \vec{0}_W$, $\vec{0}_W \in \mathrm{im}(T)$.
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$$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2).$$
Since $\vec{v}_1 + \vec{v}_2 \in V$, $\vec{w}_1 + \vec{w}_2 \in \mathrm{im}(T)$.
3. Let $\vec{w}_1 \in \mathrm{im}(V)$ and let $k \in \mathbb{R}$. Then there exists $\vec{v}_1 \in V$ such that $T(\vec{v}_1) = \vec{w}_1$, and
$$k\vec{w}_1 = kT(\vec{v}_1) = T(k\vec{v}_1).$$
Since $k\vec{v}_1 \in V$, $k\vec{w}_1 \in \mathrm{im}(T)$.

By the Subspace Test, $\mathrm{im}(T)$ is a subspace of W. ■

Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. The dimension of $\ker(T)$, $\dim(\ker(T))$ is called the **nullity** of T and is denoted $\text{nullity}(T)$, i.e.,

$$\text{nullity}(T) = \dim(\ker(T)).$$

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2. The dimension of im(T), dim(im(T)) is called the rank of T and is denoted rank (T), i.e.,

$$\text{rank}(T) = \dim(\text{im}(T)).$$

Example

If A is an $m \times n$ matrix, then

$$\mathrm{im}(T_A) = \mathrm{im}(A) = \mathrm{col}(A).$$

It follows that

$$\mathrm{rank}\,(T_A) = \dim(\mathrm{im}(T_A)) = \dim(\mathrm{col}(A)) = \mathrm{rank}\,(A).$$

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Also, $\ker(T_A) = \text{null}(A)$, so

$$\text{nullity}(T_A) = \dim(\text{null}(A)) = n - \text{rank}\,(A).$$

Finding bases of the kernel and the image

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Example (continued)

For the linear transformation T defined by $T : \mathcal{P}_1 \to \mathbb{R}$

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1,$$

we found that

$$\begin{aligned} \ker(T) &= \{ax - a \mid a \in \mathbb{R}\} \quad \text{and} \\ \text{im}(T) &= \mathbb{R}. \end{aligned}$$

Finding bases of the kernel and the image

Example (continued)

For the linear transformation T defined by $T : \mathcal{P}_1 \to \mathbb{R}$

$$T(p(x)) = p(1) \text{ for all } p(x) \in \mathcal{P}_1,$$

we found that

$$\begin{aligned} \ker(T) &= \{ax - a \mid a \in \mathbb{R}\} \quad \text{and} \\ \mathrm{im}(T) &= \mathbb{R}. \end{aligned}$$

From this, we see that $\ker(T) = \mathrm{span}\{(x-1)\}$; since $\{(x-1)\}$ is an independent subset of $\mathcal{P}_1$, $\{(x-1)\}$ is a basis of $\ker(T)$. Thus

$$\dim(\ker(T)) = 1 = \mathrm{nullity}(T).$$

Finding bases of the kernel and the image

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$$\dim(\ker(T)) = 1 = \operatorname{nullity}(T).$$

Since $\operatorname{im}(T) = \mathbb{R}$,

$$\dim(\operatorname{im}(T)) = 1 = \operatorname{rank}(T).$$

Problem

Let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Then T is a linear transformation (you should be able to prove this). Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

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Then T is a linear transformation (you should be able to prove this). Find a basis of ker(T) and a basis of im(T).

Solution

Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \ker(T)$. Then

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

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$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives us a system of four equations in the four variables a, b, c, d:

$$a+b=0; b+c=0; c+d=0; d+a=0.$$

Solution (continued)

This system has solution $a = -t, b = t, c = -t, d = t$ for any $t \in \mathbb{R}$, and thus

$$\ker(T) = \left\{ \begin{bmatrix} -t & t \\ -t & t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}.$$

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Let $B_k = \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right\}$. Since B_k is an independent subset of $\mathbf{M}_{22}$ and $\operatorname{span}(B) = \ker(T)$, B_k is a basis of $\ker(T)$.

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$$\begin{aligned} \operatorname{im}(T) &= \left\{ \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \\ &= \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}. \end{aligned}$$

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Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Solution (continued)

S is a dependent subset of $\mathbf{M}_{22}$, but (check this yourselves)

$$B_i = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

is an independent subset of S. Since $\text{span}(B_i) = \text{span}(S) = \text{im}(T)$ and B_i is independent, B_i is a basis of $\text{im}(T)$. ■

Surjections and Injections

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Definition

Let V and W be vector spaces and $T : V \to W$ a linear transformation.

1. T is onto (or surjective) if $\mathrm{im}(T) = W$.
2. T is one-to-one (or injective) if, for $\vec{v}, \vec{w} \in V$, $T(\vec{v}) = T(\vec{w})$ implies that $\vec{v} = \vec{w}$.

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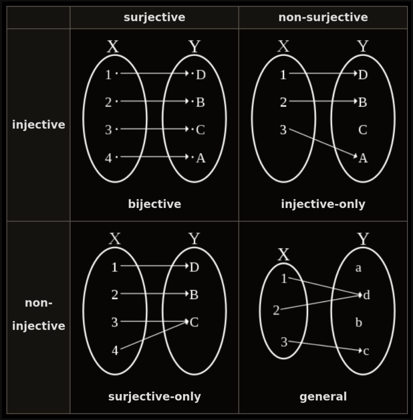
Example

Let V be a vector space. Then the identity operator on V, $1_V : V \to V$, is one-to-one and onto.


surjective
non-surjective
injective
non-
injective
X
Y
1
2
3
4
D
B
C
A
bijective
injective-only
surjective-only
a
d
b
c
general

Theorem

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Then T is one-to-one if and only if $\ker(T) = \{\vec{0}\}$.

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Proof.

($\Rightarrow$) Let $\vec{v} \in \ker(T)$. Then

$$T(\vec{v}) = \vec{0} = T(\vec{0}).$$

Since is one-to-one, $\vec{v} = \vec{0}$. But $\vec{v}$ is an arbitrary element of $\ker(T)$, and thus $\ker T = \{\vec{0}\}$.

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($\Rightarrow$) Let $\vec{v} \in \ker(T)$. Then

$$T(\vec{v}) = \vec{0} = T(\vec{0}).$$

Since is one-to-one, $\vec{v} = \vec{0}$. But $\vec{v}$ is an arbitrary element of $\ker(T)$, and thus $\ker T = \{\vec{0}\}$.

($\Leftarrow$) Conversely, suppose that $\ker(T) = \{\vec{0}\}$, and let $\vec{v}, \vec{w} \in V$ be such that

$$T(\vec{v}) = T(\vec{w}).$$

Then $T(\vec{v}) - T(\vec{w}) = \vec{0}$, and since T is a linear transformation

$$T(\vec{v} - \vec{w}) = \vec{0}.$$

By definition, $\vec{v} - \vec{w} \in \ker(T)$, implying that $\vec{v} - \vec{w} = \vec{0}$. Therefore $\vec{v} = \vec{w}$, and hence T is one-to-one. ■

Problem

Let $T : \mathbf{M}_{22} \to \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + d \\ b + c \end{bmatrix} \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}_{22}.$$

Prove that T is onto but not one-to-one.

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Prove that T is onto but not one-to-one.

Proof.

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$. Since $T \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, T is onto.

Observe that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \ker(T)$, so $\ker(T) \neq \vec{0}_{22}$. By the previous Theorem, T is not one-to-one. ■

Problem

Suppose U is an invertible $m \times m$ matrix and let $T : \mathbf{M}_{mn} \to \mathbf{M}_{mn}$ be defined by

$$T(A) = UA \text{ for all } A \in \mathbf{M}_{mn}.$$

Then T is a linear transformation (this is left to you to verify). Prove that T is one-to-one and onto.

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Then T is a linear transformation (this is left to you to verify). Prove that T is one-to-one and onto.

Proof.

Suppose $A, B \in \mathbf{M}_{mn}$ and that $T(A) = T(B)$. Then $UA = UB$; since U is invertible

$$\begin{aligned} U^{-1}(UA) &= U^{-1}(UB) \\ (U^{-1}U)A &= (U^{-1}U)B \\ I_{mm}A &= I_{mm}B \\ A &= B. \end{aligned}$$

Therefore, T is one-to-one.

Proof. (continued)

To prove that T is onto, let $B \in \mathbf{M}_{mn}$ and let $A = U^{-1}B$. Then

$$T(A) = UA = U(U^{-1}B) = (UU^{-1})B = I_{mm}B = B,$$

and therefore T is onto. ∎

Problem

Let $S : \mathcal{P}_2 \to \mathbf{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2 + bx + c \in \mathcal{P}_2.$$

Prove that S is one-to-one but not onto.

Problem

Let $S : \mathcal{P}_2 \to \mathbf{M}_{22}$ be a linear transformation defined by

$$S(ax^2 + bx + c) = \begin{bmatrix} a+b & a+c \\ b-c & b+c \end{bmatrix} \text{ for all } ax^2 + bx + c \in \mathcal{P}_2.$$

Prove that S is one-to-one but not onto.

Proof.

By definition,

$$\ker(S) = \{ax^2 + bx + c \in \mathcal{P}_2 \mid a+b=0, a+c=0, b-c=0, b+c=0\}.$$

Suppose $p(x) = ax^2 + bx + c \in \ker(S)$. This leads to a homogeneous system of four equations in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since the unique solution is $a = b = c = 0$, $\ker(S) = \{\vec{0}\}$, and thus S is one-to-one.

Proof. (continued)

To show that S is not onto, show that $\mathrm{im}(S) \neq \mathcal{P}_2$; i.e., find a matrix $A \in \mathbf{M}_{22}$ such that for every $p(x) \in \mathcal{P}_2$, $S(p(x)) \neq A$. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix},$$

and suppose $p(x) = ax^2 + bx + c \in \mathcal{P}_2$ is such that $S(p(x)) = A$. Then

$$\begin{array}{ll} a + b = 0 & a + c = 1 \\ b - c = 0 & b + c = 2 \end{array}$$

Solving this system

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right].$$

Since the system is inconsistent, there is no $p(x) \in \mathcal{P}_2$ so that $S(p(x)) = A$, and therefore S is not onto. ■

Problem (One-to-one linear transformations preserve independent sets)

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is one-to-one and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an independent subset of V, then $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is an independent subset of W.

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Proof.

Let $\vec{0}_V$ and $\vec{0}_W$ denote the zero vectors of V and W, respectively. Suppose that

$$a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) + \cdots + a_k T(\vec{v}_k) = \vec{0}_W$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Since linear transformations preserve linear combinations (addition and scalar multiplication),

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W.$$

Now, since T is one-to-one, $\ker(T) = \{\vec{0}_V\}$, and thus

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V.$$

However, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is independent, and hence $a_1 = a_2 = \cdots = a_k = 0$. Therefore, $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$ is independent. ■

Problem (Onto linear transformations preserve spanning sets)

Let V and W be vector spaces and $T : V \to W$ a linear transformation. Prove that if T is onto and $V = \mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, then

$$W = \mathrm{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

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Proof.

Suppose that T is onto and let $\mathbf{w} \in W$. Then there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Since $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, there exist $a_1, a_2, \ldots a_k \in \mathbb{R}$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$. Using the fact that T is a linear transformation,

$$\begin{aligned} \mathbf{w} = T(\mathbf{v}) &= T(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) \\ &= a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \cdots + a_kT(\mathbf{v}_k), \end{aligned}$$

i.e., $\mathbf{w} \in \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$, and thus

$$W \subseteq \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}.$$

Since $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k) \in W$, it follows that $\text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\} \subseteq W$, and therefore $W = \text{span}\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_k)\}$. ■

Suppose A is an $m \times n$ matrix. How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is onto? How do we determine if $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one?

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Theorem

Let A be an $m \times n$ matrix, and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ the linear transformation induced by A.

1. T_A is onto if and only if $\text{rank}\,(A) = m$.
2. T_A is one-to-one if and only if $\text{rank}\,(A) = n$.

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Proof. (sketch)

1. T_A is onto if and only if $\text{im}(T_A) = \mathbb{R}^m$. This is equivalent to $\text{col}(A) = \mathbb{R}^m$, which occurs if and only if $\dim(\text{col}(A)) = m$, i.e., rank $(A) = m$.
2. $\ker(T_A) = \text{null}(A)$, and $\text{null}(A) = \{\vec{0}\}$ if and only if $A\vec{x} = \vec{0}$ has the unique solution $\vec{x} = \vec{0}$. Thus and row echelon form of A has a leading one in every column, which occurs if and only if rank $(A) = n$. ∎

The Dimension Theorem (Rank-Nullity Theorem)

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Suppose A is an $m \times n$ matrix with rank r. Since $\mathrm{im}(T_A) = \mathrm{col}(A)$,

$$\dim(\mathrm{im}(T_A)) = \mathrm{rank}\,(A) = r.$$

We also know that $\ker(T_A) = \mathrm{null}(A)$, and that $\dim(\mathrm{null}(A)) = n - r$. Thus,

$$\dim(\mathrm{im}(T_A)) + \dim(\ker(T_A)) = n = \dim\ \mathbb{R}^n.$$

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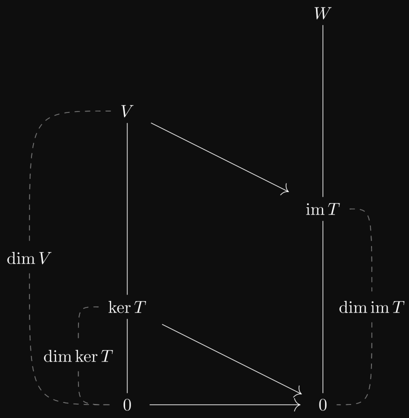
Theorem (Dimension Theorem (Rank-Nullity Theorem))

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If $\ker(T)$ and $\mathrm{im}(T)$ are both finite dimensional, then V is finite dimensional, and

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)).$$

Equivalently, $\dim(V) = \mathrm{nullity}(T) + \mathrm{rank}\,(T)$.



W
V
$\operatorname{im} T$
$\dim V$
$\ker T$
$\dim \operatorname{im} T$
$\dim \ker T$
0
0

Proof. (Outline)

Let $\vec{w} \in \text{im}(T)$; then $\vec{w} = T(\vec{v})$ for some $\vec{v} \in V$. Suppose

$$\left\{T(\vec{b}_1), T(\vec{b}_2), \ldots, T(\vec{b}_r)\right\}$$

is a basis of $\text{im}(T)$, and that

$$\left\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}$$

is a basis of $\ker(T)$. We define

$$B = \left\{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_r, \vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\right\}.$$

To prove that B is a basis of V, it remains to prove that B spans V and that B is linearly independent.

Since B is independent and spans V, B is a basis of V, implying V is finite dimensional (V is spanned by a finite set of vectors). Furthermore, $|B| = r + k$, so

$$\dim(V) = \dim(\text{im}(T)) + \dim(\ker(T)).$$

∎

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Example

Let V and W be vector spaces and $T : V \to W$ a linear transformation. If V is finite dimensional, then it follows that

$$\dim(\ker(T)) \leq \dim(V) \quad \text{and} \quad \dim(\operatorname{im}(T)) \leq \dim(V).$$

Problem

For $a \in \mathbb{R}$, recall that the linear transformation $E_a : \mathcal{P}_n \to \mathbb{R}$, the evaluation map at a, is defined as

$$E_a(p(x)) = p(a) \text{ for all } p(x) \in \mathcal{P}_n.$$

Prove that E_a is onto, and that

$$B = \{(x-a), (x-a)^2, (x-a)^3, \ldots, (x-a)^n\}$$

is a basis of $\ker(E_a)$.

Proof.

Let $t \in \mathbb{R}$, and choose $p(x) = t \in \mathcal{P}_n$. Then $p(a) = t$, so $E_a(p(x)) = t$, i.e., E_a is onto.

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By the Dimension Theorem,

$$n + 1 = \dim(\mathcal{P}_n) = \dim(\ker(E_a)) + \dim(\operatorname{im}(E_a)).$$

Since E_a is onto, $\dim(\operatorname{im}(E_a)) = \dim(\mathbb{R}) = 1$, and thus $\dim(\ker(E_a)) = n$.

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Note that $(x - a)^j \in \ker(E_a)$ for $j = 1, 2, \ldots, n$, so $B \subseteq \ker(E_a)$.

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Since $|B| = n = \dim(\ker(E_a))$, B spans $\ker(E_a)$.

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Therefore, B is a basis of $\ker(E_a)$. ■

Theorem

Let V and W be vector spaces, $T : V \to W$ a linear transformation, and

$$B = \left\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_r, \vec{b}_{r+1}, \vec{b}_{r+2}, \dots, \vec{b}_n\right\}$$

a basis of V with the property that $\left\{\vec{b}_{r+1}, \vec{b}_{r+2}, \dots, \vec{b}_n\right\}$ is a basis of $\ker(T)$. Then

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is a basis of $\mathrm{im}(T)$, and therefore $r = \mathrm{rank}\,(T)$.

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Remark (How is this useful?)

Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. If you find a basis of $\ker(T)$, then this may be used to find a basis of $\mathrm{im}(T)$: extend the basis of $\ker(T)$ to a basis of V; applying the transformation T to each of the vectors that was added to the basis of $\ker(T)$ produces a set of vectors that is a basis of $\mathrm{im}(T)$.

Problem

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and let $T : \mathbf{M}_{22} \to \mathbf{M}_{22}$ be a linear transformation defined by

$$T(X) = XA - AX \text{ for all } X \in \mathbf{M}_{22}.$$

Find a basis of $\ker(T)$ and a basis of $\operatorname{im}(T)$.

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Find a basis of ker(T) and a basis of im(T).

Solution

First note that by the Dimension Theorem,

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\mathbf{M}_{22}) = 4.$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\begin{aligned} T(X) &= AX - XA \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c & d \\ a & b \end{bmatrix} - \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \begin{bmatrix} c-b & d-a \\ a-d & b-c \end{bmatrix} \end{aligned}$$

Solution (continued)

If $X \in \ker(T)$, then $T(X) = \vec{0}_{22}$ so

$$c - b = 0, d - a = 0, a - d = 0, b - c = 0.$$

This system of four equations in four variables has general solution $a = s, b = t, c = t, d = s$ for $s, t \in \mathbb{R}$. Therefore,

$$\ker(T) = \left\{ \left[\begin{array}{cc} s & t \\ t & s \end{array} \right] \;\middle|\; s, t \in \mathbb{R} \right\} = \operatorname{span} \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}.$$

Let $B_k = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$. Since B_k is independent and spans $\ker(T)$, B_k is a basis of $\ker(T)$.

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If $X \in \ker(T)$, then $T(X) = \vec{0}_{22}$ so

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Let $B_k = \left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \right\}$. Since B_k is independent and spans $\ker(T)$, B_k is a basis of $\ker(T)$.

To find a basis of $\text{im}(T)$, extend the basis of $\ker(T)$ to a basis of $\mathbf{M}_{22}$: here is one such basis

$$\left\{ \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \right\}.$$

Solution (continued)

Thus

$$B_i = \left\{ T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

is a basis of $\mathrm{im}(T)$. ■