Math 221: LINEAR ALGEBRA

Chapter 8. Orthogonality

§8-1. Orthogonal Complements and Projections

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[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Orthogonal Bases

The Orthogonal Complement $U^{\perp}$

Definition of Orthogonal Projection

The Projection Theorem and its Implications

Projection as a Linear Transformation

Orthogonality Basis

Definition (Orthogonality)

- Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We say the $\vec{x}$ and $\vec{y}$ are orthogonal if $\vec{x} \cdot \vec{y} = 0$.
- More generally, $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is an orthogonal set if each $\vec{x}_i$ is nonzero, and every pair of distinct vectors of X is orthogonal, i.e., $\vec{x}_i \cdot \vec{x}_j = 0$ for all $i \neq j$, $1 \leq i, j \leq k$.
- A set $X = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq \mathbb{R}^n$ is an orthonormal set if X is an orthogonal set of unit vectors, i.e., $||\vec{x}_i|| = 1$ for all i, $1 \leq i \leq k$.

$$\cap$$

Definition (Linearly Independence)

Let V be a vector space and $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ a subset of V. The set S is linearly independent if the following condition holds:

$$s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k = \vec{0} \quad \Rightarrow \quad s_1 = s_2 = \cdots = s_k = 0.$$

Lemma (Independent Lemma)

Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ an independent subset of V. If $\mathbf{u}$ is a vector in V, but $\mathbf{u} \notin \text{span}(S)$, then $S' = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{u}\}$ is independent.

— v.s. —

Lemma (Orthogonal Lemma)

Suppose $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$ is an orthogonal subset of $\mathbb{R}^n$, and suppose $\vec{x} \in \mathbb{R}^n$. Define

$$\vec{f}_{m+1} = \vec{x} - \frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2}\vec{f}_m.$$

Then

1. $\vec{f}_{m+1} \cdot \vec{f}_j = 0$ for all j, $1 \leq j \leq m$.
2. If $\vec{x} \notin \text{span}\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$, then $\vec{f}_{m+1} \neq \vec{0}$, and $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m, \vec{f}_{m+1}\}$ is an orthogonal set.

Proof. (of orthogonal lemma)

(1) For any $1 \leq k \leq m$

$$
\begin{aligned}
\vec{f}_{m+1} \cdot \vec{f}_k &= \left(\vec{x} - \frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2} \vec{f}_1 - \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2} \vec{f}_2 - \cdots - \frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2} \vec{f}_m \right) \cdot \vec{f}_k \\
&= \vec{x} \cdot \vec{f}_k - \frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2} \vec{f}_1 \cdot \vec{f}_k - \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2} \vec{f}_2 \cdot \vec{f}_k - \cdots - \frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2} \vec{f}_m \cdot \vec{f}_k \\
&= \vec{x} \cdot \vec{f}_k - \frac{\vec{x} \cdot \vec{f}_k}{||\vec{f}_k||^2} \vec{f}_k \cdot \vec{f}_k \\
&= \vec{x} \cdot \vec{f}_k - \vec{x} \cdot \vec{f}_k = 0.
\end{aligned}
$$

Proof. (continued)

(2) Since $\{\vec{f}_1, \cdots, \vec{f}_m\}$ are independent, by the unique representation theorem, $\vec{x} \in \text{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$, iff there exists unique representation for $\vec{x}$

$$\vec{x} = a_1\vec{f}_1 + \cdots + a_m\vec{f}_m.$$

Using the fact that $\{\vec{f}_1, \cdots, \vec{f}_m\}$ is orthogonal, one finds that

$$a_i = \frac{\vec{x} \cdot \vec{f}_i}{||\vec{f}_i||^2}.$$

In other words,

$$\vec{x} \in \text{span}\{\vec{f}_1, \cdots, \vec{f}_m\} \quad \Longleftrightarrow \quad \vec{f}_{m+1} = \vec{x} - \frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2}\vec{f}_m = \vec{0}.$$

Now, $\vec{x} \notin \text{span}\{\vec{f}_1, \cdots, \vec{f}_m\}$ implies that $\vec{f}_{m+1} \neq \vec{0}$.

Finally, $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m, \vec{f}_{m+1}\}$ is orthogonal thanks to (1). ■

Theorem

Let U be a subspace of $\mathbb{R}^n$.

1. Every orthogonal subset $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ of U is a subset of an orthogonal basis of U.
2. U has an orthogonal basis.

Proof.

Algorithm 1: Proof of part (1) of Theorem

Input : An orthogonal set $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\} \subseteq U \subseteq \mathbb{R}^n$

```
m → n;
while span{f⃗_1, ⋯ , f⃗_n} ≠ U do
    Pick up arbitrary x⃗ ∈ U \ span{f⃗_1, ⋯ , f⃗_n};
    Let f⃗_{n+1} be given by the Orthogonal Lemma;
    Then {f⃗_1, ⋯ , f⃗_n, f⃗_{n+1}} is an orthogonal set;
    n + 1 → n;
end
```

Output: An orthogonal basis $\{\vec{f}_1, \cdots, \vec{f}_n\}$ of U

(2) If $U = \{\vec{0}\}$, done. Otherwise, find an arbitrary nonzero vector in u and run the algorithm in (1). ∎

Theorem (Gram-Schmidt Orthogonalization Algorithm)

Let U be a subset of $\mathbb{R}^n$ and let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a basis of U. Let $\vec{f}_1 = \vec{x}_1$, and for each j, $2 \leq j \leq m$, let

$$\vec{f}_j = \vec{x}_j - \frac{\vec{x}_j \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{x}_j \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{x}_j \cdot \vec{f}_{j-1}}{||\vec{f}_{j-1}||^2}\vec{f}_{j-1}.$$

Then $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ is an orthogonal basis of U, and

$$\text{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_j\} = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_j\} \quad \forall j = 1, \cdots, m.$$

Algorithm 2: Gram-Schmidt Orthogonalization Algorithm

Input : A basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\} \subseteq U \subseteq \mathbb{R}^n$

$\vec{f}_1 \leftarrow \vec{x}_1$;

for $j \leftarrow 2$ **to** m **do**

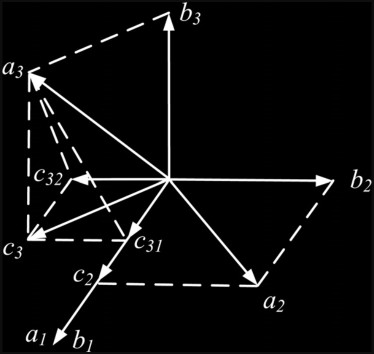
$$\vec{f}_j \leftarrow \vec{x}_j - \frac{\vec{x}_j \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{x}_j \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots - \frac{\vec{x}_j \cdot \vec{f}_{j-1}}{||\vec{f}_{j-1}||^2}\vec{f}_{j-1}.$$

end

Output: An orthogonal basis $\{\vec{f}_1, \cdots, \vec{f}_m\}$ of U s.t. $\text{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_j\} = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_j\}$ for all $j = 1, \cdots, m$.

$$\text{span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \text{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$$

basis $\rightarrow$ orthogonal basis



Problem

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. We use the Gram-Schmidt Orthogonalization Algorithm to construct an orthogonal basis B of U.

Proof.

First $\vec{f}_1 = \vec{x}_1$. Next,

$$\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally,

$$\vec{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix}.$$

Proof. (continued)

Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis of U. However, it is sometimes more convenient to deal with vectors having integer entries, in which case we take

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

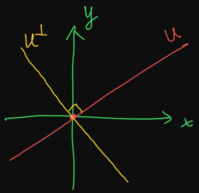
(Orthogonality of the set is not affected by multiplying vectors in the set by nonzero scalars.) ■

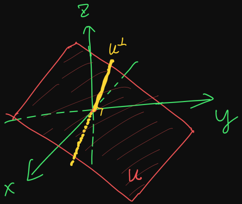
The Orthogonal Complement $U^\perp$

Definition

Let U be a subspace of $\mathbb{R}^n$. The orthogonal complement of U, called U perp, is denoted $U^\perp$ and is defined as

$$U^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y} = 0 \text{ for all } \vec{y} \in U\}.$$





Example

Let $U = \text{span}\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$, and suppose $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in U^{\perp}$. Then

$$-2a + 3b + c = 0 \quad \text{and} \quad 5a - b + 2c = 0.$$

This system of two equations in three variables has solution

$$\vec{v} = \begin{bmatrix} -7 \\ -9 \\ 13 \end{bmatrix} t, \quad \forall t \in \mathbb{R},$$

which is noting but a line passing through origin and perpendicular with the plane U.

Theorem (Properties of the Orthogonal Complement)

Let U be a subspace of $\mathbb{R}^n$.

1. $U^\perp$ is a subspace of $\mathbb{R}^n$.
2. $\{\vec{0}\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{\vec{0}\}$.
3. If $U = \text{span}\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m\}$, then

$$U^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y}_j = 0 \text{ for } j = 1, 2, \ldots, m\}.$$

Proof.

1. This is a standard subspace proof and is left as an exercise.

2. Here, $\vec{0}$ is the zero vector of $\mathbb{R}^n$. Since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\mathbb{R}^n \subseteq \{\vec{0}\}^\perp$. Since $\{\vec{0}\}^\perp \subseteq \mathbb{R}^n$, the equality follows, i.e., $\{\vec{0}\}^\perp = \mathbb{R}^n$.

   Again, since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in (\mathbb{R}^n)^\perp$, so $\{\vec{0}\} \subseteq (\mathbb{R}^n)^\perp$. Suppose $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since $\vec{x} \cdot \vec{x} = ||\vec{x}||^2$ and $\vec{x} \neq \vec{0}$, $\vec{x} \cdot \vec{x} \neq 0$, so $\vec{x} \notin (\mathbb{R}^n)^\perp$. Therefore, $\{\vec{0}\}^c \subseteq \left((\mathbb{R}^n)^\perp\right)^c$, or equivalently, $(\mathbb{R}^n)^\perp \subseteq \{\vec{0}\}$. Thus $(\mathbb{R}^n)^\perp = \{\vec{0}\}$.

Proof. (continued)

3. Let $X = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{y}_j = 0 \text{ for } j = 1, 2, \ldots, m\}$.

   "$U^\perp \subseteq X$": Suppose that $\vec{v} \in U^\perp$. Then $\vec{v}$ is orthogonal to every vector in U; in particular, $\vec{v} \cdot \vec{y}_j = 0$ for $j = 1, 2, \ldots, m$ since each such $\vec{y}_j$ is in U. Therefore, $\vec{v} \in X$. This proves that $U^\perp \subseteq X$.

   "$X \subseteq U^\perp$": Now suppose that $\vec{v} \in X$ and $\vec{u} \in U$. Then $\vec{u} = a_1\vec{y}_1 + a_2\vec{y}_2 + \cdots + a_m\vec{y}_m$ for some $a_1, a_2, \ldots, a_m \in \mathbb{R}$, and so

$$\begin{aligned} \vec{v} \cdot \vec{u} &= \vec{v} \cdot (a_1\vec{y}_1 + a_2\vec{y}_2 + \cdots + a_m\vec{y}_m) \\ &= \vec{v} \cdot (a_1\vec{y}_1) + \vec{v} \cdot (a_2\vec{y}_2) + \cdots + \vec{v} \cdot (a_m\vec{y}_m) \\ &= a_1(\vec{v} \cdot \vec{y}_1) + a_2(\vec{v} \cdot \vec{y}_2) + \cdots + a_m(\vec{v} \cdot \vec{y}_m). \end{aligned}$$

   Since $\vec{v} \in X$, $\vec{v} \cdot \vec{y}_j = 0$ for all j, $1 \leq j \leq m$. Therefore, $\vec{v} \cdot \vec{u} = 0$, and thus $X \subseteq U^\perp$.

   Finally, since $U^\perp \subseteq X$ and $X \subseteq U^\perp$, we see that $U^\perp = X$. ■

Problem

Let

$$U = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

Find $U^\perp$ by finding a basis of $U^\perp$.

Solution

$$U^\perp = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} = 0 \right\}.$$

This leads to the system of two equation in four variables

$$\begin{aligned} -b + 3c + 2d &= 0 \\ 2a + b + 4d &= 0 \end{aligned}$$

Solution (continued)

$$A = \left[\begin{array}{cccc|c} 0 & -1 & 3 & 2 & 0 \\ 2 & 1 & 0 & 4 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 3/2 & 3 & 0 \\ 0 & 1 & -3 & -2 & 0 \end{array}\right]$$

Therefore,

$$U^\perp = \left\{ \left[\begin{array}{c} -\frac{3}{2}s - 3t \\ 3s + 2t \\ s \\ t \end{array}\right] \in \mathbb{R}^4 \;\middle|\; s, t \in \mathbb{R} \right\} = \text{span}\left\{ \left[\begin{array}{c} -\frac{3}{2} \\ 3 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 3 \\ 2 \\ 0 \\ 1 \end{array}\right] \right\}.$$

Since the set $B = \left\{ \left[\begin{array}{c} -\frac{3}{2} \\ 3 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 3 \\ 2 \\ 0 \\ 1 \end{array}\right] \right\}$ is independent and spans $U^\perp$, B is a basis of $U^\perp$. ■

Remark

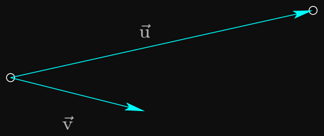
Notice that $U^\perp = \text{null}(A)$, where A is the matrix whose rows are a spanning subset of U.

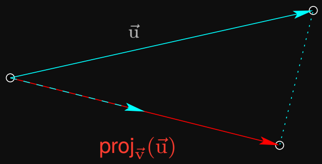
Definition of Orthogonal Projection

Theorem (Projection Formula)

Suppose $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$, $\vec{v} \neq \vec{0}$. Then the projection of $\vec{u}$ on $\vec{v}$, denoted as $\mathrm{proj}_{\vec{v}}(\vec{u})$, is equal to

$$\mathrm{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \right) \vec{v}.$$





Proof.

Let $\vec{p} = \text{proj}_{\vec{v}}(\vec{u})$; then $\vec{p}$ is parallel to $\vec{v}$, so $\vec{p} = t\vec{v}$ for some $t \in \mathbb{R}$, and $\vec{u} - \vec{p} = \vec{u} - t\vec{v}$ is orthogonal to $\vec{v}$, so

$$\begin{aligned} (\vec{u} - t\vec{v}) \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{v} - t\vec{v} \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{v} &= t||\vec{v}||^2 \end{aligned}$$

Since $\vec{v} \neq \vec{0}$,

$$t = \frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2}.$$

Therefore,

$$\vec{p} = t\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||^2} \right) \vec{v}.$$

■

Remark

Note that

- $\{\vec{v}\}$ is an orthogonal basis of the subspace U of $\mathbb{R}^3$ consisting of the line through the origin parallel to $\vec{v}$.
- $\vec{u} - \vec{p} \in U^\perp$ (since $(\vec{u} - \vec{p}) \cdot \vec{v} = 0$).

Example (Generalizing to $\mathbb{R}^n$)

Suppose U is a subspace of $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and that $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ and $\{\vec{g}_1, \vec{g}_2, \ldots, \vec{g}_m\}$ are orthogonal bases of U. Define

$$\vec{p}_f = \left(\frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2}\right)\vec{f}_1 + \left(\frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2}\right)\vec{f}_2 + \cdots + \left(\frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2}\right)\vec{f}_m \quad \text{and}$$

$$\vec{p}_g = \left(\frac{\vec{x} \cdot \vec{g}_1}{||\vec{g}_1||^2}\right)\vec{g}_1 + \left(\frac{\vec{x} \cdot \vec{g}_2}{||\vec{g}_2||^2}\right)\vec{g}_2 + \cdots + \left(\frac{\vec{x} \cdot \vec{g}_m}{||\vec{g}_m||^2}\right)\vec{g}_m.$$

Then $\vec{p}_f, \vec{p}_g \in U$ (since they are linear combinations of vectors of U) and $\vec{x} - \vec{p}_f, \vec{x} - \vec{p}_g \in U^\perp$ (by the Orthogonal Lemma). This implies that $\vec{p}_f - \vec{p}_g \in U$, and $(\vec{x} - \vec{p}_g) - (\vec{x} - \vec{p}_f) \in U^\perp$. However,

$$(\vec{x} - \vec{p}_g) - (\vec{x} - \vec{p}_f) = \vec{p}_f - \vec{p}_g,$$

and thus $\vec{p}_f - \vec{p}_g$ is in both U and $U^\perp$. This is possible if and only if $\vec{p}_f - \vec{p}_g = \vec{0}$, i.e., $\vec{p}_f = \vec{p}_g$. This means that the computation of $\vec{p}_f$ and $\vec{p}_g$ does not depend on which orthogonal basis is used.

Definition

Let $\{\vec{f}_1, \vec{f}_2, \dots, \vec{f}_m\}$ be an orthogonal basis for a subspace U of $\mathbb{R}^n$, and let $\vec{x} \in \mathbb{R}^n$. The projection of $\vec{x}$ on U is defined as

$$\text{proj}_U(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2}\right)\vec{f}_1 + \left(\frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2}\right)\vec{f}_2 + \cdots + \left(\frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2}\right)\vec{f}_m.$$

Remark

1. if $U = \{\vec{0}\}$, then $\text{proj}_{\{\vec{0}\}}(\vec{x}) = \vec{0}$ for any $\vec{x} \in \mathbb{R}^n$;
2. if $\vec{x} \in U$, then $\text{proj}_U(\vec{x})$ is also called the Fourier Expansion of $\vec{x}$.
3. In Orthogonal Lemma

$$\vec{f}_{m+1} = \vec{x} - \underbrace{\left(\frac{\vec{x} \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 + \frac{\vec{x} \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 - \cdots + \frac{\vec{x} \cdot \vec{f}_m}{||\vec{f}_m||^2}\vec{f}_m\right)}_{= \text{proj}_U(\vec{x})}.$$

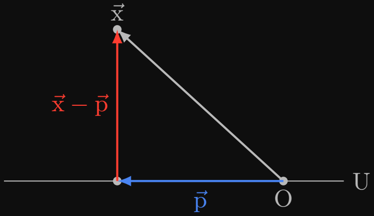
The Projection Theorem and its Implications

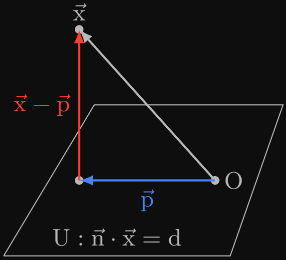
Theorem (Projection Theorem)

Let U be a subspace of $\mathbb{R}^n$, $\vec{x} \in \mathbb{R}^n$, and $\vec{p} = \mathrm{proj}_U(\vec{x})$. Then

1. $\vec{p} \in U$ and $\vec{x} - \vec{p} \in U^{\perp}$;
2. $\vec{p}$ is the vector in U closest to $\vec{x}$, meaning that for any $\vec{y} \in U$, $\vec{y} \neq \vec{p}$,

$$||\vec{x} - \vec{p}|| < ||\vec{x} - \vec{y}||.$$





Proof.

1. By definition, $\vec{p} \in U$, and by the Orthogonal Lemma, $\vec{x} - \vec{p} \in U^{\perp}$.
2. Let $\vec{y} \in U$, $\vec{y} \neq \vec{p}$. By the properties of vector addition/subtraction

$$\vec{x} - \vec{y} = (\vec{x} - \vec{p}) + (\vec{p} - \vec{y}).$$

Since $\vec{x} - \vec{p} \in U^{\perp}$ and $\vec{p} - \vec{y} \in U$,

$$(\vec{x} - \vec{p}) \cdot (\vec{p} - \vec{y}) = 0.$$

Hence, by Pythagoras' Theorem,

$$||\vec{x} - \vec{y}||^2 = ||\vec{x} - \vec{p}||^2 + ||\vec{p} - \vec{y}||^2.$$

Since $\vec{y} \neq \vec{p}$, $||\vec{p} - \vec{y}|| > 0$, so

$$||\vec{x} - \vec{y}||^2 > ||\vec{x} - \vec{p}||^2.$$

Taking square roots (since $||\vec{x} - \vec{y}||$ and $||\vec{x} - \vec{p}||$ are nonnegative),

$$||\vec{x} - \vec{y}|| > ||\vec{x} - \vec{p}||.$$

∎

Example

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}.$$

We want to find the vector in $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ closest to $\vec{v}$.

In a previous example, we used the Gram-Schmidt Orthogonalization Algorithm to construct the orthogonal basis, B, of U:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Example (continued)

By the Projection Theorem,

$$\mathrm{proj}_{\mathrm{U}}(\vec{\mathrm{v}}) = \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

is the vector in U closest to $\vec{\mathrm{v}}$.

Problem

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and let $U = \operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Find an orthogonal basis of U, and find the vector in U closest to

$$\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}.$$

Solution (Outline)

First use the Gram-Schmidt Orthogonalization Algorithm to construct an orthogonal basis of of U, and then find the projection of $\vec{v}$ on U.

Solution (continued)

Gram-Schmidt orthogonalization with

$$\begin{aligned} \vec{f}_1 &= \vec{x}_1, \\ \vec{f}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1, \\ \vec{f}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{||\vec{f}_1||^2}\vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{||\vec{f}_2||^2}\vec{f}_2 \end{aligned}$$

yields an orthogonal basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Thus the vector in U closest of $\vec{v}$ is

$$\mathrm{proj}_U(\vec{v}) = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Problem

Find the point q in the plane $3x + y - 2z = 0$ that is closest to the point $p_0 = (1, 1, 1)$.

Solution

Recall that any plane in $\mathbb{R}^3$ that contains the origin is a subspace of $\mathbb{R}^3$.

1. Find a basis X of the subspace U of $\mathbb{R}^3$ defined by the equation $3x + y - 2z = 0$.
2. Orthogonalize the basis X to get an orthogonal basis B of U.
3. Find the projection on U of the position vector of the point p_0.

Solution (continued)

1. $3x + y - 2z = 0$ is a system of one equation in three variables. Putting the augmented matrix in reduced row-echelon form

$$\left[\begin{array}{rrr|r} 3 & 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

gives general solution $x = \frac{1}{3}s + \frac{2}{3}t$, $y = s$, $z = t$ for any $s, t \in \mathbb{R}$. Then

$$U = \text{span}\left\{ \left[\begin{array}{r} -\frac{1}{3} \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} \frac{2}{3} \\ 0 \\ 1 \end{array}\right] \right\}.$$

Let

$$X = \left\{ \left[\begin{array}{r} -1 \\ 3 \\ 0 \end{array}\right], \left[\begin{array}{r} 2 \\ 0 \\ 3 \end{array}\right] \right\}$$

Then X is linearly independent and $\text{span}(X) = U$, so X is a basis of U.

Solution (continued)

1. Use the Gram-Schmidt Orthogonalization Algorithm to get an orthogonal basis of U:

$$\vec{f}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{f}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{-2}{10}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 9 \\ 3 \\ 15 \end{bmatrix}.$$

   Therefore,

$$B = \left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$$

   is an orthogonal basis of U.

Solution (continued)

3. To find the point q on U closest to $p_0 = (1,1,1)$, compute

$$\mathrm{proj}_U \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{10}\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{9}{35}\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$
$$= \frac{1}{7}\begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}.$$

Therefore, $q = \left(\frac{4}{7}, \frac{6}{7}, \frac{9}{7}\right)$. ■

Projection as a Linear Transformation

Definition

Let V and W be vector spaces, and $T : V \to W$ a linear transformation.

1. The kernel of T (sometimes called the null space of T) is defined to be the set
$$\ker(T) = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}.$$
2. The image of T is defined to be the set
$$\operatorname{im}(T) = \{T(\vec{v}) \mid \vec{v} \in V\}.$$

Theorem

Let U be a fixed subspace of $\mathbb{R}^n$, and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\vec{x}) = \operatorname{proj}_U(\vec{x}) \text{ for all } \vec{x} \in \mathbb{R}^n.$$

Then

1. T is a linear operator on $\mathbb{R}^n$;
2. $\operatorname{im}(T) = U$ and $\ker(T) = U^\perp$;
3. $\dim(U) + \dim(U^\perp) = n$.

Proof.

If $U = \{\vec{0}\}$, then $U^\perp = \mathbb{R}^n$, so $T(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$. This implies that $T = 0$ (the zero transformation), and the theorem holds.

Now suppose that $U \neq \{\vec{0}\}$. We first prove (3) based on (1) and (2):

3. Since T is a linear transformation – part (1), the Rank-Nullity Theorem implies that

$$\dim(\mathrm{im}(T)) + \dim(\ker(T)) = \dim \mathbb{R}^n = n.$$

   Applying the result from part (2), we get

$$\dim(U) + \dim(U^\perp) = n.$$

Proof. (continued)

1. Let $B = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ be an orthonormal basis of U. Then by the definition of $\text{proj}_U(\vec{x})$,

$$T(\vec{x}) = (\vec{x} \cdot \vec{f}_1)\vec{f}_1 + (\vec{x} \cdot \vec{f}_2)\vec{f}_2 + \cdots + (\vec{x} \cdot \vec{f}_m)\vec{f}_m, \qquad (1)$$

(since $\|\vec{f}_i\|^2 = 1$ for each $i = 1, 2, \ldots, m$). Let $\vec{x}, \vec{y} \in U$ and $k \in \mathbb{R}$. Then

$$\begin{aligned} T(\vec{x}+\vec{y}) &= ((\vec{x}+\vec{y}) \cdot \vec{f}_1)\vec{f}_1 + ((\vec{x}+\vec{y}) \cdot \vec{f}_2)\vec{f}_2 + \cdots + ((\vec{x}+\vec{y}) \cdot \vec{f}_m)\vec{f}_m \\ &= (\vec{x} \cdot \vec{f}_1 + \vec{y} \cdot \vec{f}_1)\vec{f}_1 + (\vec{x} \cdot \vec{f}_2 + \vec{y} \cdot \vec{f}_2)\vec{f}_2 + \\ &\quad \cdots + (\vec{x} \cdot \vec{f}_m + \vec{y} \cdot \vec{f}_m)\vec{f}_m \\ &= (\vec{x} \cdot \vec{f}_1)\vec{f}_1 + (\vec{y} \cdot \vec{f}_1)\vec{f}_1 + (\vec{x} \cdot \vec{f}_2)\vec{f}_2 + (\vec{y} \cdot \vec{f}_2)\vec{f}_2 + \\ &\quad \cdots + (\vec{x} \cdot \vec{f}_m)\vec{f}_m + (\vec{y} \cdot \vec{f}_m)\vec{f}_m \\ &= [(\vec{x} \cdot \vec{f}_1)\vec{f}_1 + (\vec{x} \cdot \vec{f}_2)\vec{f}_2 + \cdots + (\vec{x} \cdot \vec{f}_m)\vec{f}_m] \\ &\quad +[(\vec{y} \cdot \vec{f}_1)\vec{f}_1 + (\vec{y} \cdot \vec{f}_2)\vec{f}_2 + \cdots + (\vec{y} \cdot \vec{f}_m)\vec{f}_m] \\ &= T(\vec{x}) + T(\vec{y}). \end{aligned}$$

Thus $\vec{x} + \vec{y} \in U$, so T preserves addition.

Proof. (continued)

1. (continued) Also,

$$\begin{aligned} T(k\vec{x}) &= ((k\vec{x}) \cdot \vec{f}_1)\vec{f}_1 + ((k\vec{x}) \cdot \vec{f}_2)\vec{f}_2 + \cdots + ((k\vec{x}) \cdot \vec{f}_m)\vec{f}_m \\ &= (k(\vec{x} \cdot \vec{f}_1))\vec{f}_1 + (k(\vec{x} \cdot \vec{f}_2))\vec{f}_2 + \cdots + (k(\vec{x} \cdot \vec{f}_m))\vec{f}_m \\ &= k(\vec{x} \cdot \vec{f}_1)\vec{f}_1 + k(\vec{x} \cdot \vec{f}_2)\vec{f}_2 + \cdots + k(\vec{x} \cdot \vec{f}_m)\vec{f}_m \\ &= k[(\vec{x} \cdot \vec{f}_1)\vec{f}_1 + (\vec{x} \cdot \vec{f}_2)\vec{f}_2 + \cdots + (\vec{x} \cdot \vec{f}_m)\vec{f}_m] \\ &= kT(\vec{x}). \end{aligned}$$

Thus $k\vec{x} \in U$, so T preserves scalar multiplication.

Therefore, T is a linear transformation.

Proof. (continued)

2. By equation (1), $T(\vec{x}) \in U$ because $T(\vec{x})$ is a linear combination of the elements of $B \subseteq U$, and therefore $\mathrm{im}(T) \subseteq U$. Conversely, suppose that $\vec{x} \in U$. By using Fourier Expansion, $\vec{x} = T(\vec{x})$, and thus $\vec{x} \in \mathrm{im}(T)$. Therefore $U \subseteq \mathrm{im}(T)$. Since $\mathrm{im}(T) \subseteq U$ and $U \subseteq \mathrm{im}(T)$, $\mathrm{im}(T) = U$.

   To show that $\ker(T) = U^\perp$, let $\vec{x} \in U^\perp$. Then $\vec{x} \cdot \vec{f}_i = 0$ for each $i = 1, 2, \ldots, m$, so $T(\vec{x}) = \vec{0}$, implying $\vec{x} \in \ker(T)$. Thus $U^\perp \subseteq \ker(T)$. Conversely, let $\vec{x} \in \ker(T)$. Then $T(\vec{x}) = \vec{0}$, so $\vec{x} - T(\vec{x}) = \vec{x}$; but, $\vec{x} - T(\vec{x}) \in U^\perp$ (Projection Theorem), so $\vec{x} \in U^\perp$, implying that $\ker(T) \subseteq U^\perp$. Since $U^\perp \subseteq \ker(T)$ and $\ker(T) \subseteq U^\perp$, $\ker(T) = U^\perp$. ■