Math 221: LINEAR ALGEBRA

Chapter 8. Orthogonality

§8-6. Singular Value Decomposition

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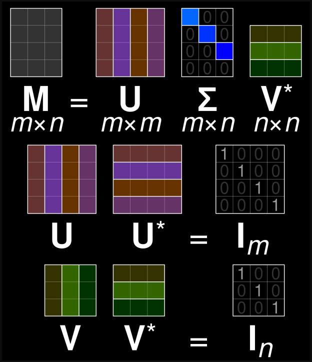
[1] Slides are adapted from those by Karen Seyffarth from University of Calgary.

Singular Value Decomposition



Definition

Let A be an $m \times n$ matrix. The singular values of A are the square roots of the nonzero eigenvalues of $A^T A$. Singular Value Decomposition (SVD) can be thought of as a generalization of orthogonal diagonalization of a symmetric matrix to an arbitrary $m \times n$ matrix.
Given an $m \times n$ matrix A, we will see how to express A as a product

$$A = U \Sigma V^T$$

where

- U is an $m \times m$ orthogonal matrix whose columns are eigenvectors of AA^T.
- V is an $n \times n$ orthogonal matrix whose columns are eigenvectors of $A^T A$.
- Σ is an $m \times n$ matrix whose only nonzero values lie on its main diagonal, and are the square roots of the eigenvalues of both AA^T and $A^T A$.

Theorem

If A is an $m \times n$ matrix, then $A^T A$ and AA^T have the same nonzero eigenvalues.

Proof.

Suppose A is an $m \times n$ matrix, and suppose that λ is a nonzero eigenvalue of $A^T A$. Then there exists a nonzero vector $\vec{x} \in \mathbb{R}^n$ such that

$$(A^T A)\vec{x} = \lambda \vec{x}. \tag{1}$$

Multiplying both sides of this equation by A:

$$\begin{aligned} A(A^T A)\vec{x} &= A\lambda\vec{x} \\ (AA^T)(A\vec{x}) &= \lambda(A\vec{x}). \end{aligned}$$

Since $\lambda \neq 0$ and $\vec{x} \neq \vec{0}_n$, $\lambda\vec{x} \neq \vec{0}_n$, and thus by equation (1), $(A^T A)\vec{x} \neq \vec{0}_n$; thus $A^T(A\vec{x}) \neq \vec{0}_n$, implying that $A\vec{x} \neq \vec{0}_m$.

Therefore $A\vec{x}$ is an eigenvector of AA^T corresponding to eigenvalue λ. An analogous argument can be used to show that every nonzero eigenvalue of AA^T is an eigenvalue of $A^T A$, thus completing the proof. ■

Examples

Example

Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Then

$$AA^T = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 5 & 11 \end{bmatrix}.$$

$$A^TA = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}.$$

Example (continued)

Since AA^T is 2×2 while A^TA is 3×3, and AA^T and A^TA have the same nonzero eigenvalues, compute $c_{AA^T}(x)$ (because it's easier to compute than $c_{A^TA}(x)$).

$$\begin{aligned} c_{AA^T}(x) &= \det(xI - AA^T) = \begin{vmatrix} x-11 & -5 \\ -5 & x-11 \end{vmatrix} \\ &= (x-11)^2 - 25 \\ &= x^2 - 22x + 121 - 25 \\ &= x^2 - 22x + 96 \\ &= (x-16)(x-6). \end{aligned}$$

Therefore, the eigenvalues of AA^T are $\lambda_1 = 16$ and $\lambda_2 = 6$.

Example (continued)

The eigenvalues of A^TA are $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$, and the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{6}$. By convention, we list the eigenvalues (and corresponding singular values) in nonincreasing order (i.e., from largest to smallest).

To find the matrix V, find eigenvectors for A^TA. Since the eigenvalues of AA^T are distinct, the corresponding eigenvectors are orthogonal, and we need only normalize them.

$\lambda_1 = 16$: solve $(16I - A^TA)\vec{y}_1 = \vec{0}$.

$$\left[\begin{array}{rrr|r} 6 & -2 & -6 & 0 \\ -2 & 14 & 2 & 0 \\ -6 & 2 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } \vec{y}_1 = \left[\begin{array}{c} t \\ 0 \\ t \end{array}\right] = t\left[\begin{array}{c} 1 \\ 0 \\ 1 \end{array}\right], t \in \mathbb{R}.$$

$\lambda_2 = 6$: solve $(6I - A^TA)\vec{y}_2 = \vec{0}$.

$$\left[\begin{array}{rrr|r} -4 & -2 & -6 & 0 \\ -2 & 4 & 2 & 0 \\ -6 & 2 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } \vec{y}_2 = \left[\begin{array}{r} -s \\ -s \\ s \end{array}\right] = s\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right], s \in \mathbb{R}.$$

Example (continued)

$\lambda_3 = 0$: solve $(-A^T A)\vec{y}_3 = \vec{0}$.

$$\left[\begin{array}{rrr|r} -10 & -2 & -6 & 0 \\ -2 & -2 & 2 & 0 \\ -6 & 2 & -10 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } \vec{y}_3 = \left[\begin{array}{r} -r \\ 2r \\ r \end{array}\right] = r\left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array}\right], r \in \mathbb{R}.$$

Let

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right], \vec{v}_2 = \frac{1}{\sqrt{3}}\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right], \vec{v}_3 = \frac{1}{\sqrt{6}}\left[\begin{array}{r} -1 \\ 2 \\ 1 \end{array}\right].$$

Then

$$V = \frac{1}{\sqrt{6}}\left[\begin{array}{rrr} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & 1 \end{array}\right].$$

Also,

$$\Sigma = \left[\begin{array}{ccc} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{array}\right],$$

and we use A, V^T, and Σ to find U.

Example (continued)

Since V is orthogonal and $A = U\Sigma V^T$, it follows that $AV = U\Sigma$. Let $V = \left[\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}\right]$, and let $U = \left[\begin{array}{cc} \vec{u}_1 & \vec{u}_2 \end{array}\right]$, where $\vec{u}_1$ and $\vec{u}_2$ are the two columns of U. Then we have

$$\begin{aligned} A\left[\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}\right] &= \left[\begin{array}{cc} \vec{u}_1 & \vec{u}_2 \end{array}\right]\Sigma \\ \left[\begin{array}{ccc} A\vec{v}_1 & A\vec{v}_2 & A\vec{v}_3 \end{array}\right] &= \left[\begin{array}{ccc} \sigma_1\vec{u}_1 + 0\vec{u}_2 & 0\vec{u}_1 + \sigma_2\vec{u}_2 & 0\vec{u}_1 + 0\vec{u}_2 \end{array}\right] \\ &= \left[\begin{array}{ccc} \sigma_1\vec{u}_1 & \sigma_2\vec{u}_2 & \vec{0} \end{array}\right] \end{aligned}$$

which implies that $A\vec{v}_1 = \sigma_1\vec{u}_1 = 4\vec{u}_1$ and $A\vec{v}_2 = \sigma_2\vec{u}_2 = \sqrt{6}\vec{u}_2$. Thus,

$$\vec{u}_1 = \frac{1}{4}A\vec{v}_1 = \frac{1}{4}\left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]\frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 0 \\ 1 \end{array}\right] = \frac{1}{4\sqrt{2}}\left[\begin{array}{r} 4 \\ 4 \end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ 1 \end{array}\right],$$

and

$$\vec{u}_2 = \frac{1}{\sqrt{6}}A\vec{v}_2 = \frac{1}{\sqrt{6}}\left[\begin{array}{rrr} 1 & -1 & 3 \\ 3 & 1 & 1 \end{array}\right]\frac{1}{\sqrt{3}}\left[\begin{array}{r} -1 \\ -1 \\ 1 \end{array}\right] = \frac{1}{3\sqrt{2}}\left[\begin{array}{r} 3 \\ -3 \end{array}\right] = \frac{1}{\sqrt{2}}\left[\begin{array}{r} 1 \\ -1 \end{array}\right].$$

Example (continued)

Therefore,

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \left(\frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -1 & 2 & 1 \end{bmatrix} \right). \end{aligned}$$

Problem

Find an SVD for $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Solution

Since A is 3×1, $A^T A$ is a 1×1 matrix whose eigenvalues are easier to find than the eigenvalues of the 3×3 matrix AA^T.

$$A^T A = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}.$$

Thus $A^T A$ has eigenvalue $\lambda_1 = 9$, and the eigenvalues of AA^T are $\lambda_1 = 9$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Furthermore, A has only one singular value, $\sigma_1 = 3$.

To find the matrix V, find an eigenvector for $A^T A$ and normalize it. In this case, finding a unit eigenvector is trivial: $\vec{v}_1 = \begin{bmatrix} 1 \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 \end{bmatrix}.$$

Solution (continued)

Also, $\Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and we use A, V^T, and Σ to find U.

Now $AV = U\Sigma$, with $V = \begin{bmatrix} \vec{v}_1 \end{bmatrix}$, and $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$, where $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ are the columns of U. Thus

$$\begin{aligned} A\begin{bmatrix} \vec{v}_1 \end{bmatrix} &= \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}\Sigma \\ \begin{bmatrix} A\vec{v}_1 \end{bmatrix} &= \begin{bmatrix} \sigma_1\vec{u}_1 + 0\vec{u}_2 + 0\vec{u}_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1\vec{u}_1 \end{bmatrix} \end{aligned}$$

This gives us $A\vec{v}_1 = \sigma_1\vec{u}_1 = 3\vec{u}_1$, so

$$\vec{u}_1 = \frac{1}{3}A\vec{v}_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Solution (continued)

The vectors $\vec{u}_2$ and $\vec{u}_3$ are eigenvectors of AA^T corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 0$. Instead of solving the system $(0I - AA^T)\vec{x} = \vec{0}$ and then using the Gram-Schmidt orthogonalization algorithm on the resulting set of two basic eigenvectors, the following approach may be used.

Find vectors $\vec{u}_2$ and $\vec{u}_3$ by first extending $\{\vec{u}_1\}$ to a basis of $\mathbb{R}^3$, then using the Gram-Schmidt algorithm to orthogonalize the basis, and finally normalizing the vectors.

Starting with $\{3\vec{u}_1\}$ instead of $\{\vec{u}_1\}$ makes the arithmetic a bit easier. It is easy to verify that

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Set

$$\vec{f}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and apply the Gram-Schmidt orthogonalization algorithm to $\{\vec{f}_1, \vec{x}_2, \vec{x}_3\}$.

Solution (continued)

This gives us

$$\vec{f}_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{f}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\vec{u}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$U = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally,

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}.$$

■

Problem

Find a singular value decomposition of $A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \right) \begin{bmatrix} \sqrt{85} & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{17}} \begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix} \right).$$

■

Remark

Since there is only one non-zero eigenvalue, $\vec{u}_2$ (the second column of U) can not be found using the formula $\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2$. However, $\vec{u}_2$ can be chosen to be any unit vector orthogonal to $\vec{u}_1$; in this case, $\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Problem

Find a singular value decomposition of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Solution

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

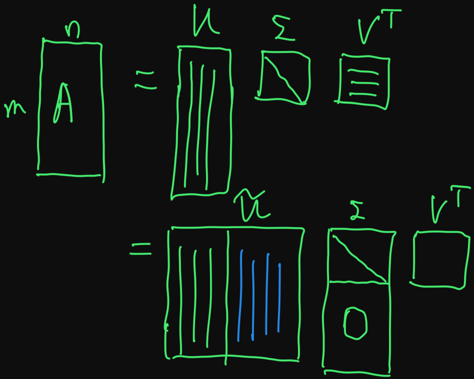
$$\|$$

$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right)\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 & 1 \\ -\sqrt{3} & 0 & \sqrt{3} \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}\right)$$

■

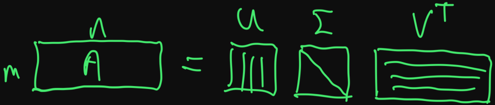
Fundamental Subspaces

Full Singular Value Decomposition



Full

Singular Value Decomposition



Applications

Example (Polar Decomposition)

$$a + bi = \underbrace{\sqrt{a^2 + b^2}}_{\text{radius}} \underbrace{e^{i\theta}}_{\text{rotation}} .$$

Similarly, any square matrix

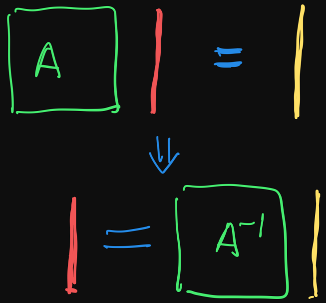
$$A = U\Sigma V^T = \underbrace{U\Sigma U^T}_{\text{nonneg. def.}} \underbrace{UV^T}_{\text{rotation}}$$

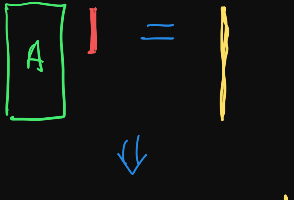
Definition

A real $n \times n$ matrix G is nonnegative definite (or positive in the book) if it is symmetric and for all $\vec{x} \in \mathbb{R}^n$

$$\vec{x}^T G \vec{x} \geq 0.$$

Example (Generalized inverse)





Example (Image of unit ball under linear transform A)

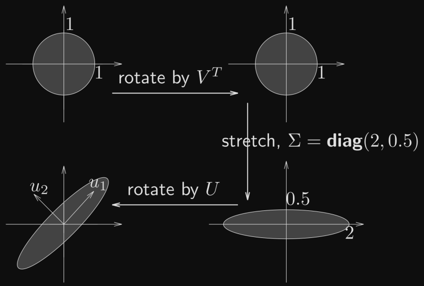
Let $A = U\Sigma V^T$ be the full SVD for an $m \times n$ matrix A. We will see how the unit ball will be mapped:

$$\{A\vec{x} \mid ||\vec{x}|| \leq 1\}$$

The linear map $\vec{y} = A\vec{x}$ is trying to do the following things:

1. Rotate the n-vector $\vec{x}$ by V^T
2. Stretch along axes by σ_i with $\sigma_i = 0$ for $i > \text{rank}(A)$
3. Zero-pad for tall matrix (i.e., $m > n$) or truncate for fat matrix (i.e., $m < n >$) to get m-vector
4. Rotate the m-vector by U^T

Example (Image of unit ball under linear transform A – continued)



Example (Image Compression)

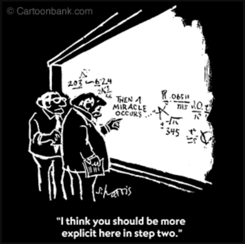


Image is a A is a 300×300 matrix.

$$A \approx \sum_{i=1}^{n} \sigma_i \vec{u}_i \vec{v}_i^T$$