Math 362: Mathmatical Statistics II

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Chapter 6. Hypothesis Testing

Plan

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Chapter 6. Hypothesis Testing

§ 6.1 Introduction

Instead of numerical estimates of parameters, in the form of either single points or confidence intervals, we want to make a choice between two conflicting theories, or **hypothesis**:

1. H_0: the null hypothesis

   v.s.

2. H_1: the null hypothesis

Comments: Hypothesis testing and confidence intervals are dual concepts to each other: one can obtained from the other. However, it is often difficult to specify μ_0 to the null hypothesis.

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§ 6.2 The Decision Rule

Go over the example first....

- **Test statistic:** Any function of the observed data whose numerical value dictates whether H_0 is accepted or rejected.

- **Critical region** C: The set of values for the test statistic that result in the null hypothesis being rejected.

  Critical value: The particular point in C that separates the rejection region from the acceptance region.

- **Level of significance** α: The probability that the test statistic lies in the critical region C under H_0.

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Test Normal mean $H_0 : \mu = \mu_0$ (σ known)

Setup:

1. Let $Y_1 = y_1, \cdots, Y_n = y_n$ be a random sample of size n from $N(\mu, \sigma^2)$ with σ known.
2. Set $\bar{y} = \frac{1}{n}(y_1 + \cdots + y_n)$ and $z = \frac{\bar{y} - \mu_0}{\sigma / \sqrt{n}}$.
3. The level of significance is α.

Test:

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases}$$

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Definition. The **P-value** associated with an observed test statistic is the probability of getting a value for that test statistic as extreme as or more extreme than what was actually observed (relative to H_1) given that H_0 is true.

Note: Test statistics that yield small P-values should be interpreted as evidence against H_0.

E.g. Suppose that test statistic $z = 0.60$. Find P-value for

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$\mathbb{P}(Z \geq 0.60) = 0.2743.$

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Setup: Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size n from Bernoulli(p). $X = \sum_{i=1}^{n} X_i \sim$ Binomial(n, p). We want to test $H_0 : p = p_0$.

1. When n is large, use Z score. Large-sample test
2. Otherwise, use the exact binomial distribution. Small-sample test

$$n \text{ is large}$$

$$\Updownarrow$$

$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$

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$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

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$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases} \qquad \begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases} \qquad \begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$. reject H_0 if $z \leq -z_\alpha$. reject H_0 if $|z| \leq z_{\alpha/2}$.

Large-sample test for p

Setup:

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size n from Bernoulli(p).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is α.

Test:

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject H_0 if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

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Small-sample test for p

E.g. $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

☐

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$\square$

Binomial with n = 19 and p = 0.85

x	P(X = x)	
6	0.000000	
7	0.000002	
8	0.000018	
9	0.000123	→ $P(X \leq 13) = 0.053696$
10	0.000699	
11	0.003242	
12	0.012246	
13	0.037366	
14	0.090746	
15	0.171409	
16	0.242829	
17	0.242829	
18	0.152892	
19	0.045599	→ $P(X = 19) = 0.045599$

Plan

§ 6.1 Introduction

§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Chapter 6. Hypothesis Testing

mi8ouk§ 6.4 Type I and Type II Errors

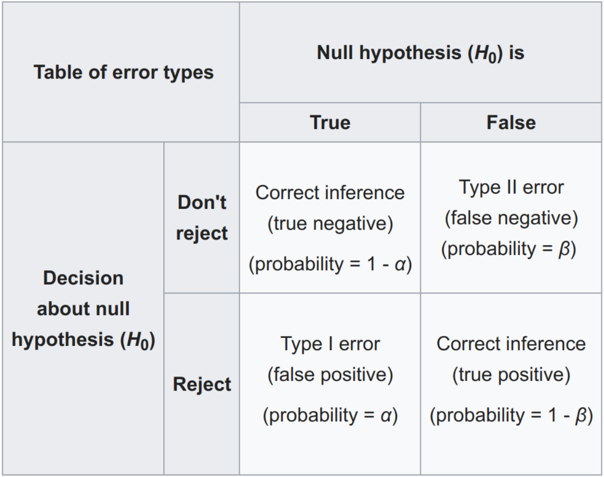

Table of error types		**Null hypothesis (H_0) is**	
		True	**False**
Decision about null hypothesis (H_0)	**Don't reject**	Correct inference (true negative) (probability = 1 - α)	Type II error (false negative) (probability = β)
	Reject	Type I error (false positive) (probability = α)	Correct inference (true positive) (probability = 1 - β)

Type I error $\sim \alpha$

$$\alpha := \mathbb{P}(\text{Type I error}) = \mathbb{P}(\text{Reject } H_0 | H_0 \text{ is true})$$

By convention, H_0 is always of the form, e.g., $\mu = \mu_0$. So this probability can be exactly determined. It is equal to the level of significance α.

(Simple null test)

Type I error $\sim \alpha$

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(Simple null test)

Type II error $\sim \beta$

$$\beta := \mathbb{P}(\text{Type II error}) = \mathbb{P}(\text{Fail to reject } H_0 | H_1 \text{ is true})$$

In order to compute Type II error, we need to specify a concrete alternative hypothesis.

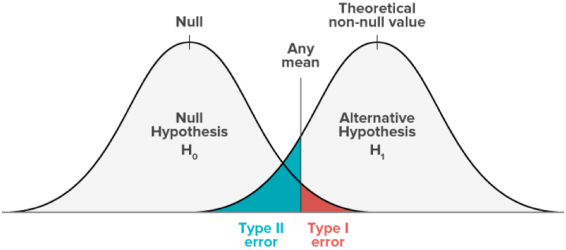

Figure: One-sided inference $H_1 : \mu > \mu_0$

Type II error $\sim \beta$

$$\beta := \mathbb{P}(\text{Type II error}) = \mathbb{P}(\text{Fail to reject } H_0 | H_1 \text{ is true})$$

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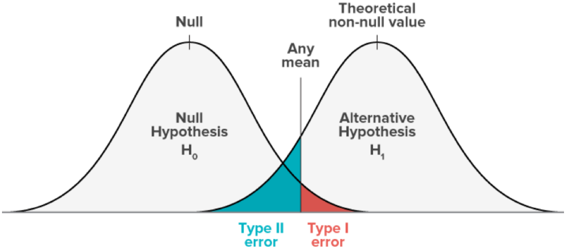


Figure: One-sided inference $H_1 : \mu > \mu_0$

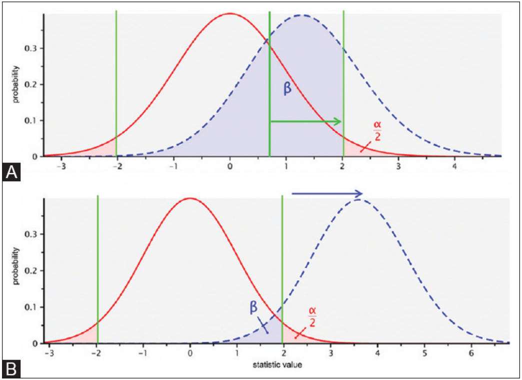
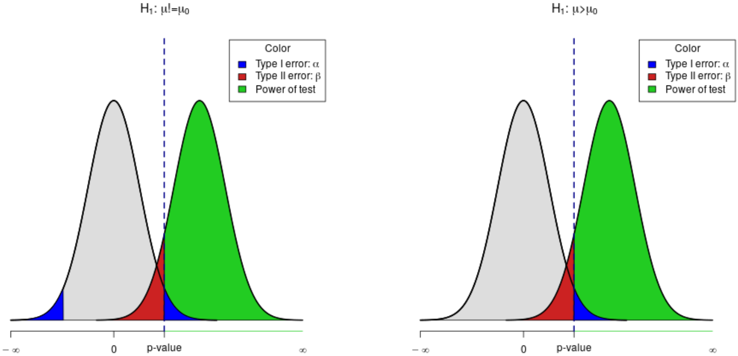


Figure: Two-sided inference $H_1 : \mu \neq \mu_0$

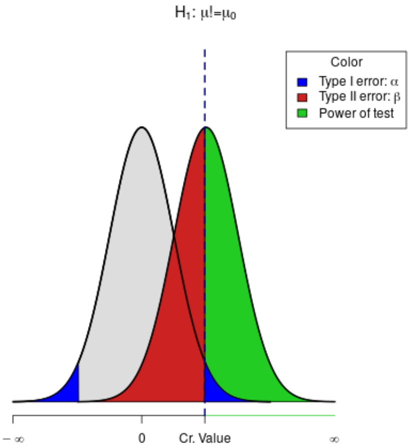
Power of test $1-\beta$

$$\text{Power of test} = \mathbb{P}(\text{Reject } H_0 | H_1 \text{ is true}) = 1-\beta$$

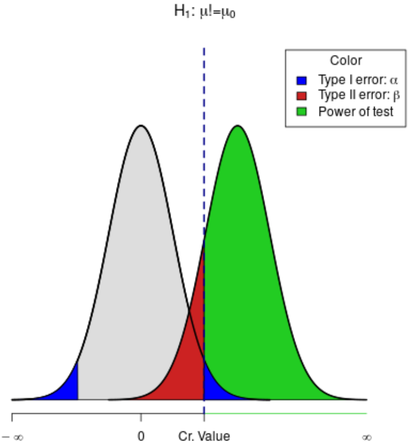


One online interactive show all α, β and $1-\beta$:
`https://rpsychologist.com/d3/NHST/`

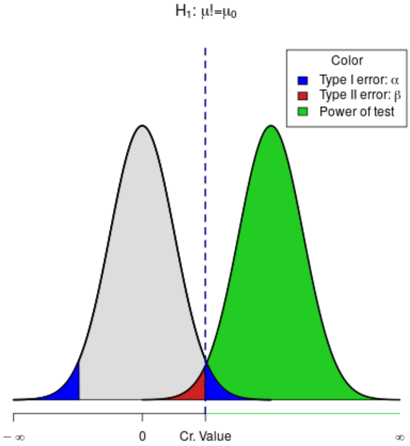
Two-sided test



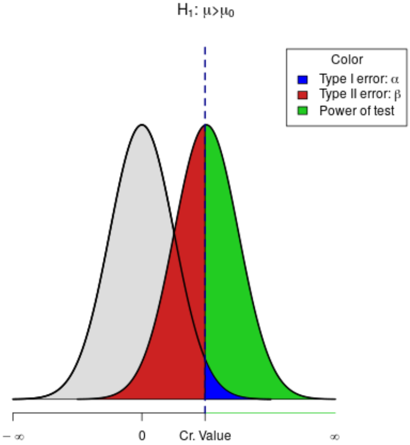
Two-sided test



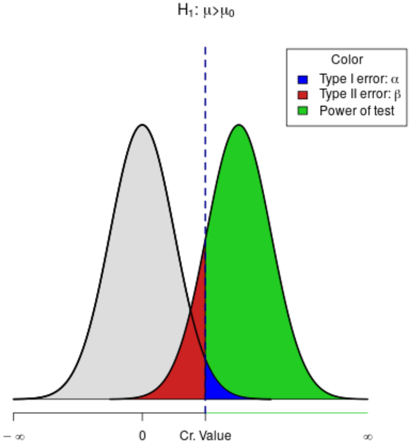
Two-sided test



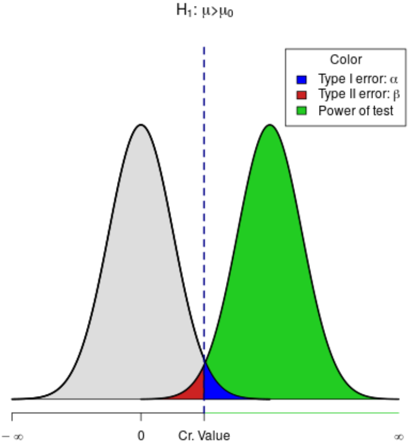
One-sided test



One-sided test



One-sided test

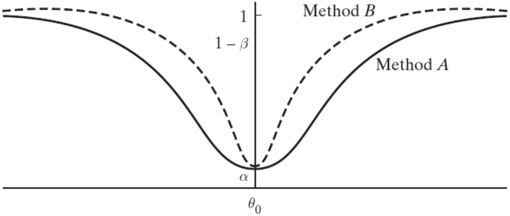


```
PlotErrorFigure <− function(shift = 3.33, TwoSided = TRUE, FileName ) {

 png(FileName) # PNG File to save the plot.
 x <− seq(−4, 4, length=1000)
 hx <− dnorm(x, mean=0, sd=1)

 if (TwoSided){ # Determine the tile of the plot
    Title <− expression(paste(H['1'],": ", mu,"!=",  mu['0']) )
 } else {
    Title <− expression(paste(H['1'],": ", mu,">",  mu['0']) )
 }

 plot(x, hx, type="n", xlim=c(−4, 8), ylim=c(0, 0.5),  ylab = "", xlab = "",
       main= Title, axes=FALSE)
 axis(1, at = c(−qnorm(.025), 0, −4),
       labels = expression("p−value", 0, −infinity ))

 # shift = qnorm(1−0.025, mean=0, sd=1)∗1.7
 xfit2 <− x + shift
 yfit2 <− dnorm(xfit2, mean=shift, sd=1)

 # Print null hypothesis area
 col_null = "#DDDDDD"
 polygon(c(min(x), x, max(x)), c(0,hx,0), col=col_null)
 lines(x, hx, lwd=2)

 # The alternative hypothesis area
 ## The red − underpowered area
 lb <− min(xfit2)
 ub <− round(qnorm(.975),2)
```

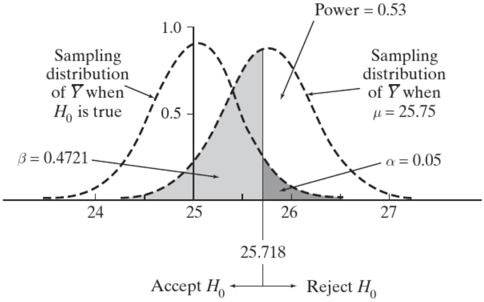
```
col1 = "#CC2222"
i <− xfit2 >= lb & xfit2 <= ub
polygon(c(lb, xfit2 [ i ], ub), c(0, yfit2 [ i ],0) , col=col1)

## The green area where the power is
col2 = "#22CC22"
i <− xfit2 >= ub
polygon(c(ub, xfit2 [ i ], max(xfit2)) , c(0, yfit2 [ i ],0) , col=col2)

# Outline the alternative hypothesis
lines ( xfit2 , yfit2 , lwd=2)
  axis(1, at = (c(ub, max(xfit2))) , labels=c("", expression( infinity )) ,
      col=col2, lwd=1, lwd. tick =FALSE)

# Now draw the type I error.
## The right part.
lines(x, hx, lwd=2)
i <− x >= ub
polygon(c(ub,x[i ], max(x)), c(0,hx[i ],0) , col="blue")
## The left part in case of two sided test.
if (TwoSided){
  i <− x <= −ub
  polygon(c(min(x),x[i],−ub), c(0,hx[i ],0) , col="blue")
}

# Line at the P−value
abline(v=ub, lwd=2, col="#000088", lty="dashed")

# Put legend
legend("topright", inset=.02, title ="Color",
        c(expression(paste("Type I error: ", alpha)),
          expression(paste("Type II error: ", beta)),
          "Power of test"),
```

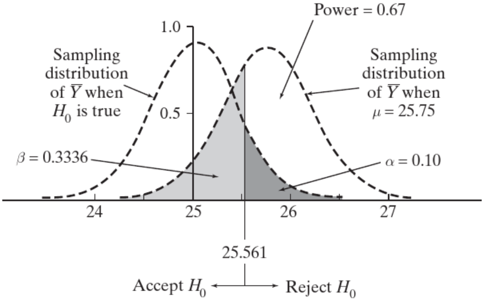
```
           fill =c("blue", col1, col2), horiz=FALSE)
  dev.off()
}

PlotErrorFigure(3,TRUE, "Type-I-II-TwoSided-3.png")
PlotErrorFigure(3,FALSE, "Type-I-II-OneSided-3.png")

PlotErrorFigure(2,TRUE, "Type-I-II-TwoSided-2.png")
PlotErrorFigure(2,FALSE, "Type-I-II-OneSided-2.png")

PlotErrorFigure(4,TRUE, "Type-I-II-TwoSided-4.png")
PlotErrorFigure(4,FALSE, "Type-I-II-OneSided-4.png")
```

Use the **power curves** to select methods
(steepest one!)

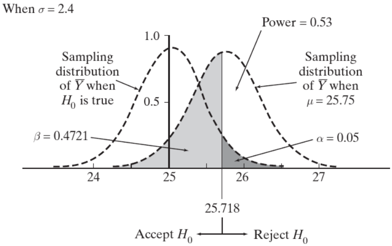


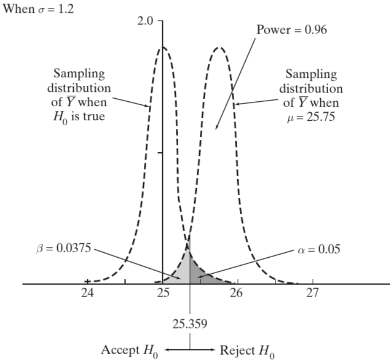
$\alpha \uparrow \quad \Longrightarrow \quad \beta \downarrow$ and $(1 - \beta) \uparrow$





$$\sigma \downarrow \quad \Longrightarrow \quad \beta \downarrow \quad \text{and} \quad (1-\beta) \uparrow$$





One usually cannot control the given parameter σ. But one can achieve the same power of test by increasing the sample size *n*.

E.g. Test $H_0 : \mu = 100$ v.s. $H_1 : \mu > 100$ at $\alpha = 0.05$ with $\sigma = 14$ known.
Requirement: $1 - \beta = 0.60$ when $\mu = 103$.
Find smallest sample size *n*.

Remark: Two condisions: $\alpha = 0.05$ and $1 - \beta = 0.60$
Two unknowns: Critical value y^* and sample size *n*

Sol.

$$C = \left\{ z : z = \frac{\bar{y} - \mu_0}{\sigma / \sqrt{n}} \geq z_\alpha \right\}.$$

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$$
\begin{aligned}
1-\beta &= \mathbb{P}\left(\frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(\frac{\overline{Y}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(Z \geq -\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} + z_\alpha \,\middle|\, \mu_1\right) \\
&= \Phi\left(\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha\right)
\end{aligned}
$$

$$
\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha = \Phi^{-1}(1-\beta) \quad \Longleftrightarrow \quad n = \left(\sigma \times \frac{\Phi^{-1}(1-\beta)+z_\alpha}{\mu_1-\mu_0}\right)^2
$$

$$
n = \left\lceil \left(14 \times \frac{0.2533+1.645}{103-100}\right)^2 \right\rceil = \lceil 78.48 \rceil = 79.
$$

☐

R: $z_\alpha = \text{qnorm}(1-\alpha)$ and $\Phi^{-1}(1-\beta) = \text{qnorm}(1-\beta)$

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$$

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$$

$$
n = \left\lceil \left(14 \times \frac{0.2533+1.645}{103-100}\right)^2 \right\rceil = \lceil 78.48 \rceil = 79.
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\end{aligned}
$$

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\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha = \Phi^{-1}(1-\beta) \quad \Longleftrightarrow \quad n = \left(\sigma \times \frac{\Phi^{-1}(1-\beta)+z_\alpha}{\mu_1-\mu_0}\right)^2
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n = \left\lceil \left(14 \times \frac{0.2533+1.645}{103-100}\right)^2 \right\rceil = \lceil 78.48 \rceil = 79.
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$\square$

R: z_α = qnorm($1-\alpha$) and $\Phi^{-1}(1-\beta)$ = qnorm($1-\beta$)

$$\begin{aligned}
1-\beta &= \mathbb{P}\left(\frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(\frac{\overline{Y}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(Z \geq -\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} + z_\alpha \,\middle|\, \mu_1\right) \\
&= \Phi\left(\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha\right)
\end{aligned}$$

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Nonnormal data

Test $H_0 : \theta = \theta_0$, with $f_Y(y;\theta)$ is not normal distribution.

1. Identify a sufficient estimator $\widehat{\theta}$ for θ
2. Find the critical region C: Least compatible with H_0 but
   still admissible under H_1
3. Given $\alpha \to$ find $C \to \beta$, $1-\beta$...

   From $C \to$ determine α

   From $\theta_e \to$ find P-value

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Examples for nonnormal data

E.g. 1. A random sample of size n from uniform distr. $f_Y(y;\theta) = 1/\theta$, $y \in [0,\theta]$. To test

$$H_0 : \theta = 2.0 \quad \text{v.s.} \quad H_1 : \theta < 2.0$$

at the level $\alpha = 0.10$ of significance, one can use the decision rule based on Y_{max}. Find the probability of committing a Type II error when $\theta = 1.7$.

Remark: Y_{max} is a sufficient estimator for θ. Why?

Sol. 1) The critical region should has the form: $C = \{y_{max} : y_{max} \le c\}$.

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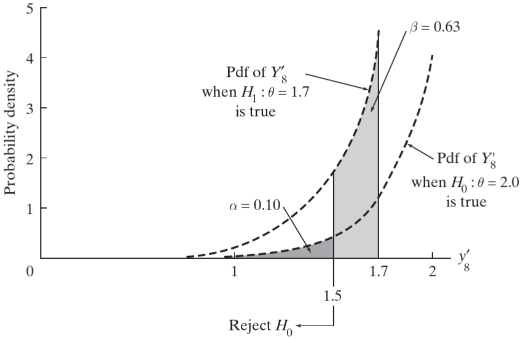

$$f_{Y_{max}}(y) = ... = n\frac{y^{n-1}}{\theta^n} \quad y \in [0, \theta].$$

$$\alpha = \int_0^c n\frac{y^{n-1}}{\theta_0^n}\mathrm{d}y = \left(\frac{c}{\theta_0}\right)^n \quad \Longrightarrow \quad c = \theta_0\alpha^{1/n} \qquad \text{(Under } H_0 : \theta = \theta_0\text{)}$$

$$\beta = \int_{\theta_0\alpha^{1/n}}^1 n\frac{y^{n-1}}{\theta_1^n}\mathrm{d}y = 1 - \left(\frac{\theta_0}{\theta_1}\right)^n \alpha \qquad \text{(Under } \theta = \theta_1\text{)}$$

E.g. 2. A random sample of size 4 from Poisson(λ): $p_X(k;\lambda) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \cdots$. One wants to test

$$H_0 : \lambda = 0.8 \quad \text{v.s.} \quad H_1 : \lambda > 0.8.$$

at the level $\alpha = 0.10$. Find power of test when $\lambda = 1.2$.

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$\overline{X}$ ~Poisson(3.2)

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4) Alternative $\lambda = 1.2 \to 1 - \beta = 0.35$.

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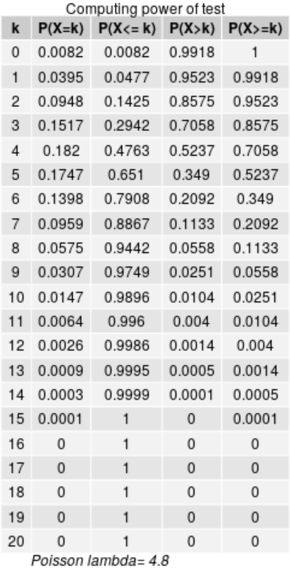
Finding critical region

k	P(X=k)	P(X<= k)	P(X>k)	P(X>=k)
0	0.0408	0.0408	0.9592	1
1	0.1304	0.1712	0.8288	0.9592
2	0.2087	0.3799	0.6201	0.8288
3	0.2226	0.6025	0.3975	0.6201
4	0.1781	0.7806	0.2194	0.3975
5	0.114	0.8946	0.1054	0.2194
6	0.0608	0.9554	0.0446	0.1054
7	0.0278	0.9832	0.0168	0.0446
8	0.0111	0.9943	0.0057	0.0168
9	0.004	0.9982	0.0018	0.0057
10	0.0013	0.9995	0.0005	0.0018
11	0.0004	0.9999	0.0001	0.0005
12	0.0001	1	0	0.0001
13	0	1	0	0
14	0	1	0	0

Poisson lambda= 3.2



Computing power of test

k	P(X=k)	P(X<= k)	P(X>k)	P(X>=k)
0	0.0082	0.0082	0.9918	1
1	0.0395	0.0477	0.9523	0.9918
2	0.0948	0.1425	0.8575	0.9523
3	0.1517	0.2942	0.7058	0.8575
4	0.182	0.4763	0.5237	0.7058
5	0.1747	0.651	0.349	0.5237
6	0.1398	0.7908	0.2092	0.349
7	0.0959	0.8867	0.1133	0.2092
8	0.0575	0.9442	0.0558	0.1133
9	0.0307	0.9749	0.0251	0.0558
10	0.0147	0.9896	0.0104	0.0251
11	0.0064	0.996	0.004	0.0104
12	0.0026	0.9986	0.0014	0.004
13	0.0009	0.9995	0.0005	0.0014
14	0.0003	0.9999	0.0001	0.0005
15	0.0001	1	0	0.0001
16	0	1	0	0
17	0	1	0	0
18	0	1	0	0
19	0	1	0	0
20	0	1	0	0

Poisson lambda= 4.8

$$1-\beta = \mathbb{P}\,(\text{Reject } H_0 \mid H_1 \text{ is true}) = \mathbb{P}(\overline{X} \geq 6|\overline{X} \sim \mathit{Poisson}(4.8))$$

```r
PlotPoissonTable <- function(n=14,lambda=3.2,png_filename,TableTitle) {
  library(gridExtra)
  library(grid)
  library(gtable)
  x = seq(1,n,1)
  # qpois(0.90,lambda)
  tb = cbind(x,
             round(dpois(x,lambda),4),
             round(ppois(x,lambda),4),
             round(1-ppois(x,lambda),4),
             round(c(1,(1-ppois(x,lambda))[1:n]),4))
  colnames(tb) <- c("k", "P(X=k)", "P(X<= k)", "P(X>k)", "P(X>=k)")
  rownames(tb) <-x
  table <- tableGrob(tb,rows = NULL)
  title <- textGrob(TableTitle,gp=gpar(fontsize=12))
  footnote <- textGrob(paste("Poisson lambda=",lambda),
                       x=0, hjust=0, gp=gpar( fontface="italic"))
  padding <- unit(0.2,"line")
  table <- gtable_add_rows(table, heights = grobHeight(title) + padding,pos = 0)
  table <- gtable_add_rows(table, heights = grobHeight(footnote)+ padding)
  table <- gtable_add_grob(table, list( title , footnote),
                           t=c(1, nrow(table)), l=c(1,2), r=ncol(table))
  png(png_filename)
  grid.draw(table)
  dev.off()
}

PlotPoissonTable(14,3.2,"Example_6-4-3_1.png","Finding critical region")
PlotPoissonTable(20,4.8,"Example_6-4-3_2.png","Computing power of test")
```

E.g. 3. A random sample of size 7 from $f_Y(y;\theta) = (\theta+1)y^{\theta}$, $y \in [0,1]$. Test

$$H_0 : \theta = 2.0 \quad \text{v.s.} \quad H_1 : \theta > 2.0$$

Decision rule: Let X be the number of y_i's that exceed 0.9 and reject H_0 if $X \geq 4$.

Find α.

Sol. 1) $X \sim \text{binomial}(7, p)$.

2) Find p:

$$\begin{aligned} p &= \mathbb{P}(Y \geq 0.9 | H_0 \text{ is true}) \\ &= \int_{0.9}^{1} 3y^2 dy = 0.271 \end{aligned}$$

3) Compute α:

$$\alpha = \mathbb{P}(X \geq 4 | \theta = 2) = \sum_{k=4}^{7} \binom{7}{k} 0.271^k 0.729^{7-k} = 0.092.$$

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§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.4 Type I and Type II Errors

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

Chapter 6. Hypothesis Testing

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Question:

Vector parameter	Scale parameter
Composite-vs-composite test	Simple-vs-composite test
$H_0 : \theta \in \omega$ vs $H_1 : \theta \in \Omega \cap \omega^c$	$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

E.g. Two normal populations $N(\mu_i, \sigma_i)$, $i = 1, 2$. σ_i are known, μ_i unknown.

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§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

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- Let $Y_1, \cdots, Y_n$ be a random sample of size n from $f_Y(y; \theta_1, \cdots, \theta_k)$
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$$C = \{\lambda : \lambda \in (0, \lambda^*]\}$$

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   In practice, little effect on this change.

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Sol. 1) Simple null hypothesis test $\Longrightarrow$

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Find a test statistic Λ for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Sol. Since the null hypothesis is simple, we have that

$$L(\omega_e) = L(p_0) = \prod_{i=1}^{n} (1-p_0)^{k_i-1} p_0 = (1-p_0)^{n\bar{k}-n} p_0^n,$$

which shows that $\bar{k}$ is a sufficient estimator.

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