Math 362: Mathmatical Statistics II

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Chapter 7. Inference Based on The Normal Distribution

Plan

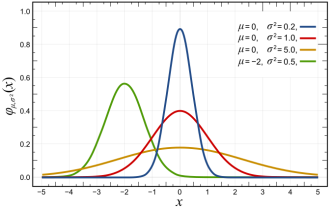
§ 7.1 Introduction

Carl Friedrich Gauss discovered the normal distribution in 1809 as a way to rationalize the method of least squares.

(1777-1855)

Marquis de Laplace proved the central limit theorem in 1810, consolidating the importance of the normal distribution in statistics.

(1749-1827)



Notation	$\mathcal{N}(\mu,\sigma^2)$
Parameters	$\mu \in \mathbb{R}$ = mean (location) $\sigma^2 > 0$ = variance (squared scale)
Support	$x \in \mathbb{R}$
PDF	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
CDF	$\frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$
Quantile	$\mu+\sigma\sqrt{2}\operatorname{erf}^{-1}(2p-1)$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
MAD	$\sigma\sqrt{2/\pi}$
Skewness	0
Ex. kurtosis	0
Entropy	$\frac{1}{2}\log(2\pi e\sigma^2)$
MGF	$\exp(\mu t+\sigma^2t^2/2)$
CF	$\exp(i\mu t-\sigma^2t^2/2)$
Fisher information	$\mathcal{I}(\mu,\sigma)=\begin{pmatrix}1/\sigma^2 & 0\\ 0 & 2/\sigma^2\end{pmatrix}$ $\mathcal{I}(\mu,\sigma^2)=\begin{pmatrix}1/\sigma^2 & 0\\ 0 & 1/(2\sigma^4)\end{pmatrix}$
Kullback-Leibler divergence	$D_{\mathrm{KL}}(\mathcal{N}_0 \parallel \mathcal{N}_1)=\frac{1}{2}\left\{\left(\frac{\sigma_0}{\sigma_1}\right)^2+\frac{(\mu_1-\mu_0)^2}{\sigma_1^2}-1+2\ln\frac{\sigma_1}{\sigma_0}\right\}$

https://en.wikipedia.org/wiki/Normal_distribution

Test for normal parameters (one sample test)

Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$.

Prob. 1 Find a test statistic Λ in order to test $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$.

When σ^2 is known: $\Lambda = \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$

When σ^2 is unknown: $\Lambda = ?$ $\quad \Lambda \stackrel{?}{=} \dfrac{\overline{Y} - \mu_0}{s/\sqrt{n}} \sim ?$

Prob. 2 Find a test statistic Λ in order to test $H_0 : \sigma^2 = \sigma_0^2$ v.s. $H_1 : \sigma^2 \neq \sigma_0^2$.

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Sol. Composite-vs-composite test with:

$$\omega = \left\{(\mu, \sigma^2) : \mu = \mu_0,\ \sigma^2 > 0\right\}$$

$$\Omega = \left\{(\mu, \sigma^2) : \mu \in \mathbb{R},\ \sigma^2 > 0\right\}$$

The MLE under the two spaces are:

$$\omega_e = (\mu_e, \sigma_e^2): \qquad \mu_e = \mu_0 \quad \text{and} \quad \sigma_e^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu_0)^2 \qquad (\text{Under } \omega)$$

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$$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i - \mu}{\sigma}\right)^2\right)$$

$$L(\omega_e) = \cdots = \left[\frac{ne^{-1}}{2\pi\sum_{i=1}^{n}(y_i - \mu_0)^2}\right]^{n/2}$$

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Hence

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left[\frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{\sum_{i=1}^{n}(y_i - \mu_0)^2}\right]^{n/2} = \cdots = \left[1 + \frac{n(\bar{y} - \mu_0)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}\right]^{-n/2}$$

$$= \left[1 + \frac{1}{n-1}\left(\frac{\bar{y} - \mu_0}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2} \Big/ \sqrt{n}}\right)^2\right]^{-n/2}$$

$$= \left[1 + \frac{1}{n-1}\left(\frac{\bar{y} - \mu_0}{s \,/\, \sqrt{n}}\right)^2\right]^{-n/2}$$

$$= \left[1 + \frac{t^2}{n-1}\right]^{-n/2}, \quad t = \frac{\bar{y} - \mu_0}{s \,/\, \sqrt{n}}$$

Finally, the test statistic is

$$\boxed{T = \frac{\overline{Y} - \mu_0}{S/\sqrt{n}}}$$

$$\text{with} \quad \overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2.$$

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$$= \left[\frac{\frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2}{\frac{n}{n-1}\sigma_0^2}\right]^{n/2} \exp\left(-\frac{n-1}{2\sigma_0^2}\frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2 + \frac{n}{2}\right)$$

$$= \left[\frac{s^2}{\frac{n}{n-1}\sigma_0^2}\right]^{n/2} \exp\left(-\frac{n-1}{2\sigma_0^2}s^2 + \frac{n}{2}\right)$$

Finally, the test statistic is

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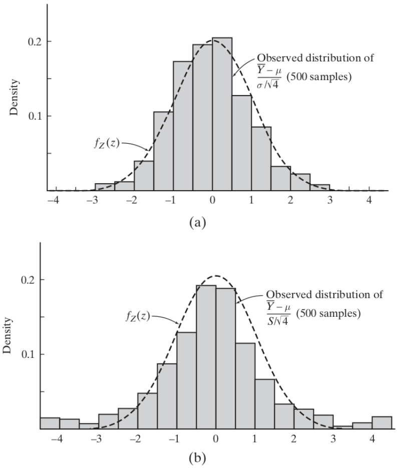
§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$



Statistician William Sealy Gosset, known as "Student"

Ref. Student’s t distribution comes from William Sealy Gosset’s 1908 paper in Biometrika under the pseudonym "Student".

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§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

Def. **Sampling distributions**

Distributions of $\underbrace{\textit{functions of random sample}}_{\text{statistics / estimators}}$ of given size.

E.g. A random sample of size n from $N(\mu, \sigma^2)$ with σ^2 known.

Sample mean $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i \sim N(\mu, \sigma^2/n)$

Aim: Determine distributions for

Sample variance $S^2 := \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$ *Chi square distr.*

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Thm 1. Let $U = \sum_{i=1}^{m} Z_j^2$, where Z_j are independent $N(0,1)$ normal r.v.s. Then

$$U \sim \text{Gamma(shape}=m/2\text{,rate}=1/2).$$

namely,

$$f_U(u) = \frac{1}{2^{m/2}\Gamma(m/2)} u^{\frac{m}{2}-1} e^{-u/2}, \qquad u \geq 0$$

Def 1. U in Thm 1 is called **chi square distribution** with m dgs of freedom.

Thm 1. Let $U = \sum_{i=1}^{m} Z_j^2$, where Z_j are independent $N(0,1)$ normal r.v.s. Then

$$U \sim \text{Gamma(shape=}m/2\text{,rate=}1/2\text{)}.$$

namely,

$$f_U(u) = \frac{1}{2^{m/2}\Gamma(m/2)} u^{\frac{m}{2}-1} e^{-u/2}, \qquad u \geq 0$$

Def 1. U in Thm 1 is called **chi square distribution** with m dgs of freedom.

Proof First take $m = 1$. For any $u \geq 0$,

$$F_{Z^2}(u) = P(Z^2 \leq u) = P\left(-\sqrt{u} \leq Z \leq \sqrt{u}\right) = 2P\left(0 \leq Z \leq \sqrt{u}\right)$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{u}} e^{-z^2/2}\, dz$$

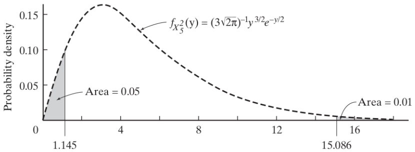
Differentiating both sides of the equation for $F_{Z^2}(u)$ gives $f_{Z^2}(u)$:

$$f_{Z^2}(u) = \frac{d}{du} F_{Z^2}(u) = \frac{2}{\sqrt{2\pi}} \frac{1}{2\sqrt{u}} e^{-u/2} = \frac{1}{2^{1/2}\Gamma\left(\frac{1}{2}\right)} u^{(1/2)-1} e^{-u/2}$$

Notice that $f_U(u) = f_{Z^2}(u)$ has the form of a gamma pdf with $r = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. By Theorem 4.6.4, then, the sum of m such squares has the stated gamma distribution with $r = m\left(\frac{1}{2}\right) = \frac{m}{2}$ and $\lambda = \frac{1}{2}$. □

Chi Square Table

	p							
df	.01	.025	.05	.10	.90	.95	.975	.99
1	0.000157	0.000982	0.00393	0.0158	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	6.304	18.549	21.026	23.336	26.217



$$\mathbb{P}(\chi^2_5 \leq 1.145) = 0.05 \iff \chi^2_{0.05,5} = 1.145$$

$$\mathbb{P}(\chi^2_5 \leq 15.086) = 0.99 \iff \chi^2_{0.99,5} = 15.086$$

```
> pchisq(1.145, df = 5)
[1] 0.04995622
> pchisq(15.086, df = 5)
[1] 0.9899989
```

```
> qchisq(0.05, df = 5)
[1] 1.145476
> qchisq(0.99, df = 5)
[1] 15.08627
```

Thm 2. Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

(a) S^2 and $\overline{Y}$ are independent.

(b) $\dfrac{(n-1)S^2}{\sigma^2} = \dfrac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 \quad \sim \quad \text{Chi Square}(n-1)$.

Proof. We will prove the case $n = 2$.

$$\overline{Y} = \frac{Y_1 + Y_2}{2}, \qquad Y_1 - \overline{Y} = \frac{Y_1 - Y_2}{2}, \qquad Y_2 - \overline{Y} = \frac{Y_2 - Y_1}{2}$$

$$S^2 = \ldots = \frac{1}{2}(Y_1 - Y_2)^2$$

(a) It is equivalanet to show $Y_1 + Y_2 \perp Y_1 - Y_2$. Since they are normal, it suffices to show that

$$\mathbb{E}[(Y_1 + Y_2)(Y_1 - Y_2)] = \mathbb{E}[Y_1 + Y_2]\mathbb{E}[Y_1 - Y_2]$$

(b) $\frac{(n-1)S^2}{\sigma^2} = \left(\frac{Y_1 - Y_2}{\sqrt{2}\sigma}\right)^2$ and $\frac{Y_1 - Y_2}{\sqrt{2}\sigma} \sim N(0, 1)$ ... $\square$

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Def 2. If $U \sim \text{Chi Square}(n)$ and $V \sim \text{Chi Square}(m)$, and $U \perp V$, then

$$F := \frac{V/m}{U/n}$$

follows the **(Snedecor's) F distribution** with m and n degrees of freedom.

Thm 3. Let $F_{m,n} = \frac{V/m}{U/n}$ be an F r.v. with m and n degrees of freedom. Then

$$f_{F_{m,n}}(w) = \frac{\Gamma\left(\frac{m+n}{2}\right) m^{m/2} n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \times \frac{w^{m/2-1}}{(n+mw)^{(m+n)/2}}, \quad w \geq 0$$

Equivalently,

$$f_{F_{m,n}}(w) = B(m/2, n/2)^{-1} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n} w\right)^{-\frac{m+n}{2}}$$

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Recall

Theorem 3.8.4 *Let X and Y be independent continuous random variables, with pdfs $f_X(x)$ and $f_Y(y)$, respectively. Assume that X is zero for at most a set of isolated points. Let $W = Y/X$. Then*

$$f_W(w) = \int_{-\infty}^{\infty} |x|\, f_X(x)\, f_Y(wx)\, dx$$

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Proof of Thm 3.

□

Proof We begin by finding the pdf for V/U. From Theorem 7.3.1 we know that $f_V(v) = \frac{1}{2^{m/2}\Gamma(m/2)} v^{(m/2)-1} e^{-v/2}$ and $f_U(u) = \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2}$.

From Theorem 3.8.4, we have that the pdf of $W = V/U$ is

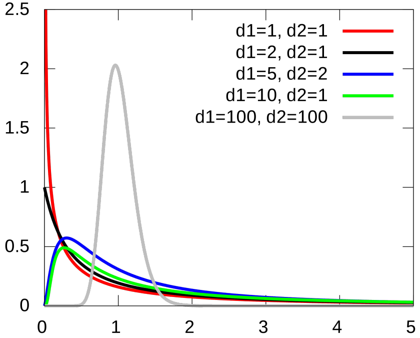
$$
\begin{aligned}
f_{V/U}(w) &= \int_0^\infty |u| f_U(u) f_V(uw)\, du \\
&= \int_0^\infty u \frac{1}{2^{n/2}\Gamma(n/2)} u^{(n/2)-1} e^{-u/2} \frac{1}{2^{m/2}\Gamma(m/2)} (uw)^{(m/2)-1} e^{-uw/2}\, du \\
&= \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \int_0^\infty u^{\frac{n+m}{2}-1} e^{-[(1+w)/2]u}\, du
\end{aligned}
$$

The integrand is the variable part of a gamma density with $r = (n+m)/2$ and $\lambda = (1+w)/2$. Thus, the integral equals the inverse of the density's constant. This gives

$$
f_{V/U} = \frac{1}{2^{(n+m)/2}\Gamma(n/2)\Gamma(m/2)} w^{(m/2)-1} \frac{\Gamma\left(\frac{n+m}{2}\right)}{[(1+w)/2]^{\frac{n+m}{2}}} = \frac{\Gamma\left(\frac{n+m}{2}\right)}{\Gamma(n/2)\Gamma(m/2)} \frac{w^{(m/2)-1}}{(1+w)^{\frac{n+m}{2}}}
$$

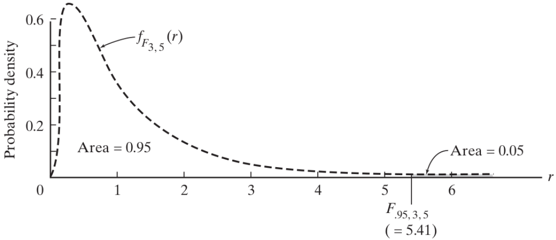
The statement of the theorem, then, follows from Theorem 3.8.2:

$$
f_{\frac{V/m}{U/n}}(w) = f_{\frac{n}{m}V/U}(w) = \frac{1}{n/m} f_{V/U}\left(\frac{w}{n/m}\right) = \frac{m}{n} f_{V/U}\left(\frac{m}{n} w\right) \qquad \square
$$




```r
# Draw F density
x=seq(0,5,0.01)
pdf= cbind(df(x, df1 = 1, df2 = 1),
df(x, df1 = 2, df2 = 1),
df(x, df1 = 5, df2 = 2),
df(x, df1 = 10, df2 = 1),
df(x, df1 = 100, df2 = 100))
matplot(x,pdf, type = "l")
 title ("F with various dgrs of freedom")
```

F Table



$$\mathbb{P}(F_{3,5} \leq 5.41) = 0.95 \quad \iff \quad F_{0.95,3,5} = 5.41$$

```
> pf(5.41,df1 = 3, df2 = 5)
[1] 0.9500093
```

```
> qf(0.95,df1 = 3, df2 = 5)
[1] 5.409451
```

Def 3. Suppose $Z \sim N(0,1)$, $U \sim$ Chi Square(n), and $Z \perp U$. Then

$$T_n = \frac{Z}{\sqrt{U/n}}$$

follows the **Student's t-distribution** of n degrees of freedom.

Remark $T_n^2 \sim F$-distribution with 1 and n degrees of freedom.

Thm 4. The pdf of the Student t of degree n is

$$f_{T_n}(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \times \left(1 + \frac{t^2}{n}\right)^{-\frac{n+2}{2}}, \quad t \in \mathbb{R}.$$

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Proof Note that $T_n^2 = \frac{Z^2}{U/n}$ has an F distribution with 1 and n df. Therefore,

$$f_{T_n^2}(t) = \frac{n^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} t^{-1/2} \frac{1}{(n+t)^{(n+1)/2}}, \quad t > 0$$

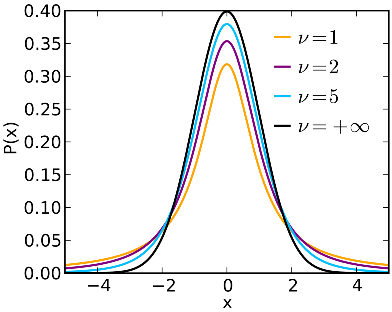
Suppose that $t > 0$. By the symmetry of $f_{T_n}(t)$,

$$\begin{aligned} F_{T_n}(t) = P(T_n \le t) &= \frac{1}{2} + P(0 \le T_n \le t) \\ &= \frac{1}{2} + \frac{1}{2} P(-t \le T_n \le t) \\ &= \frac{1}{2} + \frac{1}{2} P\left(0 \le T_n^2 \le t^2\right) \\ &= \frac{1}{2} + \frac{1}{2} F_{T_n^2}(t^2) \end{aligned}$$

Differentiating $F_{T_n}(t)$ gives the stated result:

$$\begin{aligned} f_{T_n}(t) = F'_{T_n}(t) &= t \cdot f_{T_n^2}(t^2) \\ &= t \frac{n^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} (t^2)^{-(1/2)} \frac{1}{(n+t^2)^{(n+1)/2}} \\ &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{\left[1 + \left(\frac{t^2}{n}\right)\right]^{(n+1)/2}} \end{aligned}$$

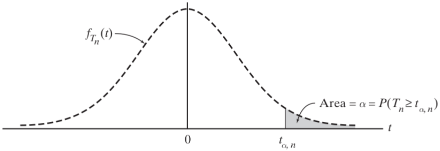
□




```r
# Draw Student t-density
x=seq(-5,5,0.01)
pdf= cbind(dt(x, df = 1),
           dt(x, df = 2),
           dt(x, df = 5),
           dt(x, df = 100))
matplot(x,pdf, type = "l")
 title ("Student's t-distributions")
```

t Table

	α						
df	.20	.15	.10	.05	.025	.01	.005
1	1.376	1.963	3.078	6.3138	12.706	31.821	63.657
2	1.061	1.386	1.886	2.9200	4.3027	6.965	9.9248
3	0.978	1.250	1.638	2.3534	3.1825	4.541	5.8409
4	0.941	1.190	1.533	2.1318	2.7764	3.747	4.6041
5	0.920	1.156	1.476	2.0150	2.5706	3.365	4.0321
6	0.906	1.134	1.440	1.9432	2.4469	3.143	3.7074
⋮			⋮				
30	0.854	1.055	1.310	1.6973	2.0423	2.457	2.7500
∞	0.84	1.04	1.28	1.64	1.96	2.33	2.58



$$\mathbb{P}(T_3 > 4.541) = 0.01 \quad \Longleftrightarrow \quad t_{0.01,3} = 4.541$$

```
> 1-pt(4.541, df =3)
[1]  0.009998238
```

```
> alpha = 0.01
> qt(1-alpha, df = 3)
[1]  4.540703
```

Thm 5. Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Then

$$T_{n-1} = \frac{\overline{Y} - \mu}{S/\sqrt{n}} \quad \sim \quad \text{Student's t of degree } n-1.$$

Proof.

$$\frac{\overline{Y} - \mu}{S/\sqrt{n}} = \frac{\dfrac{\overline{Y} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2(n-1)}}}$$

$$\frac{\overline{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1) \qquad \perp \qquad \frac{(n-1)S^2}{\sigma^2} \sim \text{Chi Square}(n-1)$$

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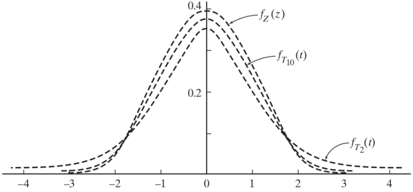

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Proof By Stirling's formula:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z (1 + O(1/z)) \qquad \text{as } z \to \infty$$

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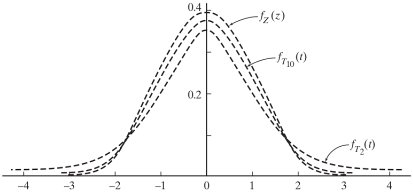

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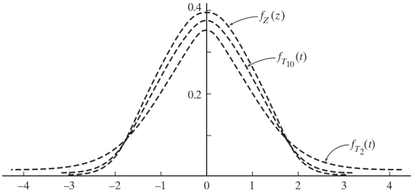

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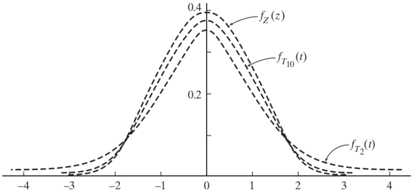

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Plan

§ 7.1 Introduction

§ 7.2 Comparing $\frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$ and $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.3 Deriving the Distribution of $\frac{\overline{Y}-\mu}{S/\sqrt{n}}$

§ 7.4 Drawing Inferences About μ

§ 7.5 Drawing Inferences About σ^2

§ 7.4 Drawing Inferences About μ

Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, \sigma^2)$.

Question Find a test statistic Λ in order to test $H_0 : \mu = \mu_0$ v.s. $H_1 : \mu \neq \mu_0$.

Case I. σ^2 is known: $\Lambda = \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n}}$

Case II. σ^2 is unknown: $\Lambda = ?$ $\Lambda \overset{?}{=} \dfrac{\overline{Y} - \mu_0}{s/\sqrt{n}} \sim ?$

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Summary

A random sample of size n from
a normal distribution $N(\mu, \sigma^2)$

	σ^2 known	σ^2 unknown
Statistic	$Z = \frac{\overline{Y}-\mu}{\sigma/\sqrt{n}}$	$T_{n-1} = \frac{\overline{Y}-\mu}{S/\sqrt{n}}$
Score	$z = \frac{\bar{y}-\mu}{\sigma/\sqrt{n}}$	$t = \frac{\bar{y}-\mu}{s/\sqrt{n}}$
Table	z_α	$t_{\alpha,n-1}$
$100(1-\alpha)$% C.I.	$\left(\bar{y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$	$\left(\bar{y} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \bar{y} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right)$
Test $H_0 : \mu = \mu_0$		
$H_1 : \mu > \mu_0$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $t \geq t_{\alpha,n-1}$
$H_1 : \mu < \mu_0$	Reject H_0 if $z \leq z_\alpha$	Reject H_0 if $t \leq t_{\alpha,n-1}$
$H_1 : \mu \neq \mu_0$	Reject H_0 if $\|z\| \geq z_{\alpha/2}$	Reject H_0 if $\|t\| \geq t_{\alpha/2,n-1}$

Computing s from data

Step 1 $a = \sum_{i=1}^{n} y_i$

Step 2. $b = \sum_{i=1}^{n} y_i^2$

Step 3. $s = \sqrt{\dfrac{nb - a^2}{n(n-1)}}$

Proof.

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 = \frac{n\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} y_i\right)^2}{n(n-1)}$$

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E.g. 1 How far apart are the bat and the insect when the bat first senses that insect is there?

Or, what is the effective range of a bat's echolocation system?

Table 7.4.1

Catch Number	Detection Distance (cm)
1	62
2	52
3	68
4	23
5	34
6	45
7	27
8	42
9	83
10	56
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Answer the question by contruct a 95% C.I.

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E.g. 2 Bank approval rates for inner-city residents v.s. rural ones.

Approval rate for rurual residents is 62%.

Do bank treat two groups equally? $\alpha = 0.05$

Table 7.4.3

Bank	Location	Affiliation	Percent Approved
1	3rd & Morgan	AU	59
2	Jefferson Pike	TU	65
3	East 150th & Clark	TU	69
4	Midway Mall	FT	53
5	N. Charter Highway	FT	60
6	Lewis & Abbot	AU	53
7	West 10th & Lorain	FT	58
8	Highway 70	FT	64
9	Parkway Northwest	AU	46
10	Lanier & Tower	TU	67
11	King & Tara Court	AU	51
12	Bluedot Corners	FT	59

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Table 7.4.4

Banks	n	$\bar{y}$	s	t Ratio	Critical Value	Reject H_0?
All	12	58.667	6.946	−1.66	±2.2010	No

Table 7.4.5

Banks	n	$\bar{y}$	s	t Ratio	Critical Value	Reject H_0?
American United	4	52.25	5.38	−3.63	±3.1825	Yes
Federal Trust	5	58.80	3.96	−1.81	±2.7764	No
Third Union	3	67.00	2.00	+4.33	±4.3027	Yes

Plan

§ 7.5 Drawing Inferences About σ^2

For a random sample of size n from $N(\mu, \sigma^2)$:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$$

$$\Downarrow$$

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2 \sim \text{Chi Square}(n-1)$$

$$\mathbb{P}\left(\chi^2_{\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right) = 1-\alpha.$$

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$100(1-\alpha)\%$ C.I. for σ:

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}}\right)$$

Testing $H_0 : \sigma^2 = \sigma_0^2$

v.s.

(at the α level of significance)

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

$H_1 : \sigma^2 < \sigma_0^2$:

Reject H_0 if

$$\chi^2 \leq \chi^2_{\alpha,n-1}$$

$H_1 : \sigma^2 \neq \sigma_0^2$:

Reject H_0 if

$\chi^2 \leq \chi^2_{\alpha/2,n-1}$ or

$$\chi^2 \geq \chi^2_{1-\alpha/2,n-1}$$

$H_1 : \sigma^2 > \sigma_0^2$:

Reject H_0 if

$$\chi^2 \geq \chi^2_{1-\alpha,n-1}$$

E.g. 1. The width of a confidence interval for σ^2 is a function of n and S^2:

$$W = \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} - \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

Find the smallest n such that the average width of a 95% C.I. for σ^2 is no greater than $0.8\sigma^2$.

Sol. Notice that $\mathbb{E}[S^2] = \sigma^2$. Hence, we need to find n s.t.

$$(n-1)\left(\frac{1}{\chi^2_{0.025,n-1}} - \frac{1}{\chi^2_{0.975,n-1}}\right) \leq 0.8.$$

Trial and error (numerics on R) gives $n = 57$.

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Trial and error (numerics on R) gives $n = 57$.

```
> # Example 7.5.1
> n=seq(45,60,1)
> l=qchisq(0.025,n-1)
> u=qchisq(0.975,n-1)
> e=(n-1)* (1/l-1/u)
> m=cbind(n,l,u,e)
> colnames(m) = c("n",
+                  "chi(0.025,n-1)",
+                  "chi(0.975,n-1)",
+                  "error")
> m
       n chi(0.025,n-1) chi(0.975,n-1)     error
 [1,] 45       27.57457       64.20146 0.9103307
 [2,] 46       28.36615       65.41016 0.8984312
 [3,] 47       29.16005       66.61653 0.8869812
 [4,] 48       29.95620       67.82065 0.8759533
 [5,] 49       30.75451       69.02259 0.8653224
 [6,] 50       31.55492       70.22241 0.8550654
 [7,] 51       32.35736       71.42020 0.8451612
 [8,] 52       33.16179       72.61599 0.8355901
 [9,] 53       33.96813       73.80986 0.8263340
[10,] 54       34.77633       75.00186 0.8173761
[11,] 55       35.58634       76.19205 0.8087008
[12,] 56       36.39811       77.38047 0.8002937
[13,] 57       37.21159       78.56716 0.7921414
[14,] 58       38.02674       79.75219 0.7842313
[15,] 59       38.84351       80.93559 0.7765517
[16,] 60       39.66186       82.11741 0.7690918
```

Case Study 7.5.2

Mutual funds are investment vehicles consisting of a portfolio of various types of investments. If such an investment is to meet annual spending needs, the owner of shares in the fund is interested in the average of the annual returns of the fund. Investors are also concerned with the volatility of the annual returns, measured by the variance or standard deviation. One common method of evaluating a mutual fund is to compare it to a benchmark, the Lipper Average being one of these. This index number is the average of returns from a universe of mutual funds.

The Global Rock Fund is a typical mutual fund, with heavy investments in international funds. It claimed to best the Lipper Average in terms of volatility over the period from 1989 through 2007. Its returns are given in the table below.

Year	Investment Return %	Year	Investment Return %
1989	15.32	1999	27.43
1990	1.62	2000	8.57
1991	28.43	2001	1.88
1992	11.91	2002	−7.96
1993	20.71	2003	35.98
1994	−2.15	2004	14.27
1995	23.29	2005	10.33
1996	15.96	2006	15.94
1997	11.12	2007	16.71
1998	0.37		

The standard deviation for these returns is 11.28%, while the corresponding figure for the Lipper Average is 11.67%. Now, clearly, the Global Rock Fund has a smaller standard deviation than the Lipper Average, but is this small difference due just to random variation? The hypothesis test is meant to answer such questions.

$$H_0 : \sigma^2 = (11.67)^2$$

versus

$$H_1 : \sigma^2 < (11.67)^2$$

Let $\alpha = 0.05$. With $n = 19$, the critical value for the chi square ratio [from part (b) of Theorem 7.5.2] is $\chi^2_{1-\alpha,n-1} = \chi^2_{.05,18} = 9.390$ (see Figure 7.5.3). But

$$\chi^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(19-1)(11.28)^2}{(11.67)^2} = 16.82$$

so our decision is clear: Do not reject H_0.