Math 362: Mathmatical Statistics II

Le Chen
`le.chen@emory.edu`

Emory University
Atlanta, GA

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Chapter 6. Hypothesis Testing

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§ 6.1 Introduction

Instead of numerical estimates of parameters, in the form of either single points or confidence intervals, we want to make a choice between two conflicting theories, or **hypothesis**:

1. H_0: the null hypothesis

   v.s.

2. H_1: the null hypothesis

Comments: Hypothesis testing and confidence intervals are dual concepts to each other: one can obtained from the other. However, it is often difficult to specify μ_0 to the null hypothesis.

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§ 6.2 The Decision Rule

§ 6.3 Testing Binomial Data – $H_0 : p = p_0$

§ 6.2 The Decision Rule

Go over the example first....

- **Test statistic:** Any function of the observed data whose numerical value dictates whether H_0 is accepted or rejected.

- **Critical region** C: The set of values for the test statistic that result in the null hypothesis being rejected.

  Critical value: The particular point in C that separates the rejection region from the acceptance region.

- **Level of significance** α: The probability that the test statistic lies in the critical region C under H_0.

Test Normal mean $H_0 : \mu = \mu_0$ (σ known)

Setup:

1. Let $Y_1 = y_1, \cdots, Y_n = y_n$ be a random sample of size n from $N(\mu, \sigma^2)$ with σ known.
2. Set $\bar{y} = \frac{1}{n}(y_1 + \cdots + y_n)$ and $z = \frac{\bar{y} - \mu_0}{\sigma / \sqrt{n}}$.
3. The level of significance is α.

Test:

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases}$$

reject H_0 if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$$

reject H_0 if $|z| \leq z_{\alpha/2}$.

- **Simple hypothesis**: Any hypothesis which specifies the population distribution completely.
- **Composite hypothesis**: Any hypothesis which does not specify the population distribution completely.

Conv. We always assume H_0 is simple and H_1 is composite.

Definition. The **P-value** associated with an observed test statistic is the probability of getting a value for that test statistic as extreme as or more extreme than what was actually observed (relative to H_1) given that H_0 is true.

Note: Test statistics that yield small P-values should be interpreted as evidence against H_0.

E.g. Suppose that test statistic $z = 0.60$. Find P-value for

$$\begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu > \mu_0 \end{cases} \qquad \begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu < \mu_0 \end{cases} \qquad \begin{cases} H_0 : \mu = \mu_0 \\ H_1 : \mu \neq \mu_0 \end{cases}$$

$\mathbb{P}(Z \geq 0.60) = 0.2743.$ $\mathbb{P}(Z \leq 0.60) = 0.7257.$

$$\mathbb{P}(|Z| \geq 0.60) \\ = 2 \times 0.7257 \\ = 0.5486.$$

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Setup: Let $X_1 = k_1, \cdots, X_n = k_n$ be a‘ random sample of size n from Bernoulli(p). $X = \sum_{i=1}^{n} X_i \sim$ Binomial(n, p). We want to test $H_0 : p = p_0$.

1. When n is large, use Z score.	Large-sample test
2. Otherwise, use the exact binomial distribution.	Small-sample test

$$n \text{ is large}$$
$$\Updownarrow$$
$$0 < np_0 - 3\sqrt{np_0(1-p_0)} < np_0 + 3\sqrt{np_0(1-p_0)} < n$$
$$\Updownarrow$$
$$n > 9 \times \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right).$$

Large-sample test for p

Setup:

1. Let $X_1 = k_1, \cdots, X_n = k_n$ be a random sample of size n from Bernoulli(p).
2. Suppose $n > 9 \max\left(\frac{1-p_0}{p_0}, \frac{p_0}{1-p_0}\right)$.
3. Set $k = k_1 + \cdots + k_n$ and $z = \frac{k-np_0}{\sqrt{np_0(1-p_0)}}$.
4. The level of significance is α.

Test:

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p > p_0 \end{cases}$$

reject H_0 if $z \geq z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p < p_0 \end{cases}$$

reject H_0 if $z \leq -z_\alpha$.

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

reject H_0 if $|z| \leq z_{\alpha/2}$.

Small-sample test for p

E.g. $n = 19$, $p_0 = 0.85$, $\alpha = 0.10$. Find critical region for the two-sided test

$$\begin{cases} H_0 : p = p_0 \\ H_1 : p \neq p_0 \end{cases}$$

Sol. $19 = n < 9 \times \max\left(\frac{0.85}{0.15}, \frac{0.15}{0.85}\right) = 51$, so small sample test.

By checking the table, the critical region is

$$C = \{k : k \leq 13 \quad \text{or} \quad k = 19\},$$

so that

$$\begin{aligned} \alpha &= \mathbb{P}(X \in C | H_0 \text{ is true}) \\ &= \mathbb{P}(X \leq 13 | p = 0.85) + \mathbb{P}(X = 19 | p = 0.85) \\ &= 0.099295 \approx 0.10. \end{aligned}$$

□

Binomial with n = 19 and p = 0.85

x	P(X = x)	
6	0.000000	$\left.\right\} \rightarrow P(X \le 13) = 0.053696$
7	0.000002	
8	0.000018	
9	0.000123	
10	0.000699	
11	0.003242	
12	0.012246	
13	0.037366	
14	0.090746	
15	0.171409	
16	0.242829	
17	0.242829	
18	0.152892	
19	0.045599	$\rightarrow P(X = 19) = 0.045599$