Math 362: Mathmatical Statistics II

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Chapter 6. Hypothesis Testing

§ 6.4 Type I and Type II Errors

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Table of error types		**Null hypothesis (H_0) is**	
		True	**False**
Decision about null hypothesis (H_0)	**Don't reject**	Correct inference (true negative) (probability = 1 - α)	Type II error (false negative) (probability = β)
	Reject	Type I error (false positive) (probability = α)	Correct inference (true positive) (probability = 1 - β)

Type I error $\sim \alpha$

$$\alpha := \mathbb{P}(\text{Type I error}) = \mathbb{P}(\text{Reject } H_0 | H_0 \text{ is true})$$

By convention, H_0 is always of the form, e.g., $\mu = \mu_0$. So this probability can be exactly determined. It is equal to the level of significance α.

(Simple null test)

Type II error $\sim \beta$

$$\beta := \mathbb{P}(\text{Type II error}) = \mathbb{P}(\text{Fail to reject } H_0 | H_1 \text{ is true})$$

In order to compute Type II error, we need to specify a concrete alternative hypothesis.

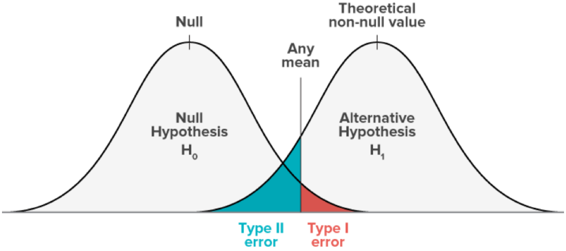


Figure: One-sided inference $H_1 : \mu > \mu_0$

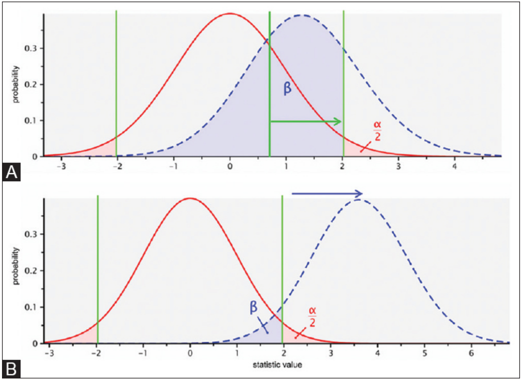
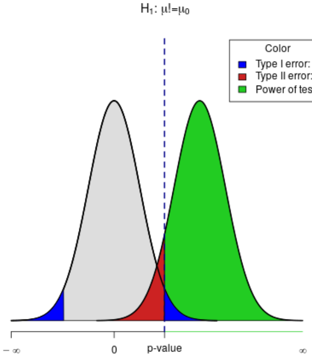
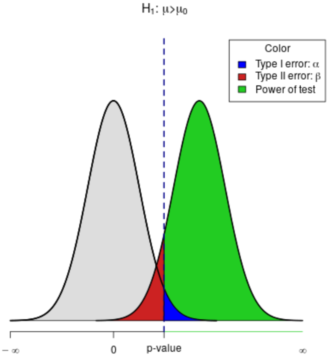


Figure: Two-sided inference $H_1 : \mu \neq \mu_0$

Power of test $1-\beta$

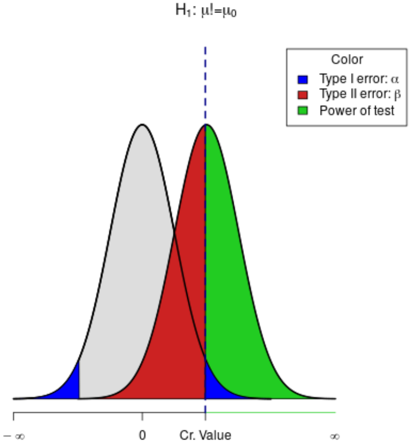
$$\text{Power of test} = \mathbb{P}(\text{Reject } H_0 | H_1 \text{ is true}) = 1-\beta$$

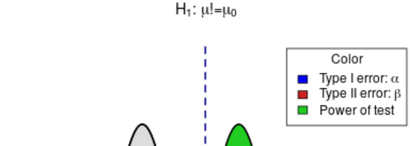




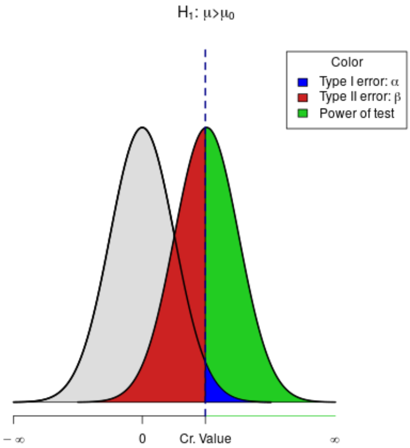
One online interactive show all α, β and $1-\beta$:
`https://rpsychologist.com/d3/NHST/`

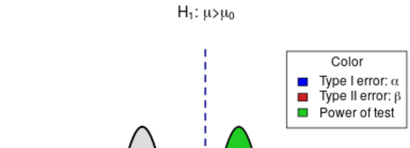
Two-sided test





One-sided test



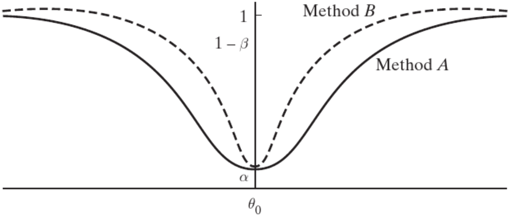


```r
PlotErrorFigure <- function(shift  = 3.33,  TwoSided = TRUE, FileName ) {

 png(FileName) # PNG File to save the plot.
 x <- seq(-4, 4, length=1000)
 hx <- dnorm(x, mean=0, sd=1)

  if  (TwoSided){ # Determine the tile  of  the  plot
    Title  <- expression(paste(H['1'],":  ",  mu,"!=",   mu['0' ]) )
 }  else  {
    Title  <- expression(paste(H['1'],":  ",  mu,">",   mu['0' ]) )
 }

 plot (x,  hx,  type="n",  xlim=c(-4, 8),  ylim=c(0,  0.5) ,    ylab = "",  xlab = "",
       main= Title,  axes=FALSE)
 axis (1,  at  = c(-qnorm(.025), 0, -4),
       labels  = expression("p-value", 0, -infinity  ))

 #  shift  = qnorm(1-0.025, mean=0, sd=1)*1.7
  xfit2  <- x + shift
  yfit2  <- dnorm(xfit2, mean=shift, sd=1)

 # Print  null  hypothesis area
 col_null  = "#DDDDDD"
 polygon(c(min(x),  x,  max(x)),  c(0,hx,0),  col=col_null)
 lines (x,  hx,  lwd=2)

 # The alternative  hypothesis area
 ## The red - underpowered area
 lb  <- min(xfit2)
 ub <- round(qnorm(.975),2)
```

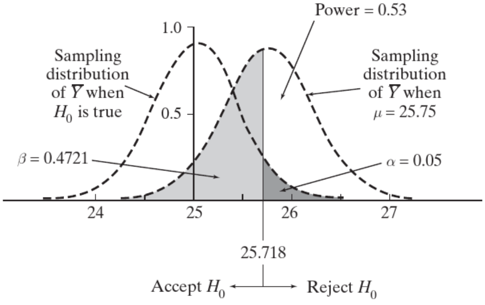
```r
col1 = "#CC2222"
i <- xfit2 >= lb & xfit2 <= ub
polygon(c(lb, xfit2[i], ub), c(0, yfit2[i],0), col=col1)

## The green area where the power is
col2 = "#22CC22"
i <- xfit2 >= ub
polygon(c(ub,xfit2[i],max(xfit2)), c(0, yfit2[i],0), col=col2)

# Outline the alternative hypothesis
lines(xfit2, yfit2, lwd=2)
  axis(1, at = (c(ub, max(xfit2))), labels=c("", expression(infinity)),
      col=col2, lwd=1, lwd.tick=FALSE)

# Now draw the type I error.
## The right part.
lines(x, hx, lwd=2)
i <- x >= ub
polygon(c(ub,x[i],max(x)), c(0,hx[i],0), col="blue")
## The left part in case of two sided test.
if (TwoSided){
  i <- x <= -ub
  polygon(c(min(x),x[i],-ub), c(0,hx[i],0), col="blue")
}

# Line at the P-value
abline(v=ub, lwd=2, col="#000088", lty="dashed")

# Put legend
legend("topright", inset=.02, title ="Color",
         c(expression(paste("Type I error: ", alpha)),
           expression(paste("Type II error: ", beta)),
           "Power of test"),
```

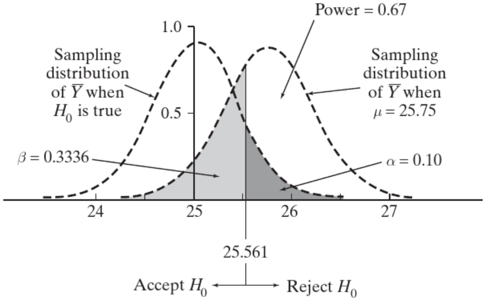
```
          fill =c("blue", col1, col2), horiz=FALSE)
  dev.off()
}

PlotErrorFigure(3,TRUE, "Type-I-II-TwoSided-3.png")
PlotErrorFigure(3,FALSE, "Type-I-II-OneSided-3.png")

PlotErrorFigure(2,TRUE, "Type-I-II-TwoSided-2.png")
PlotErrorFigure(2,FALSE, "Type-I-II-OneSided-2.png")

PlotErrorFigure(4,TRUE, "Type-I-II-TwoSided-4.png")
PlotErrorFigure(4,FALSE, "Type-I-II-OneSided-4.png")
```

Use the **power curves** to select methods
(steepest one!)

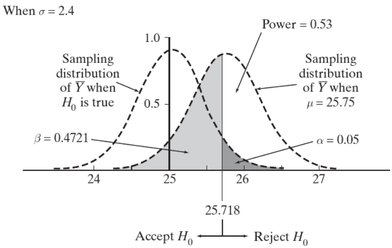


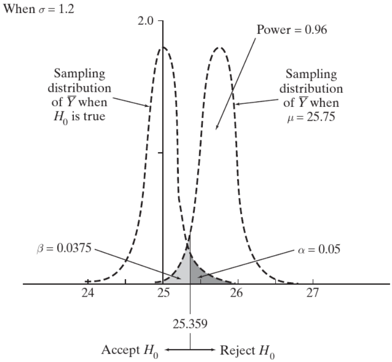
$$\alpha \uparrow \implies \beta \downarrow \text{ and } (1-\beta)\uparrow$$





$$\sigma \downarrow \implies \beta \downarrow \text{ and } (1-\beta) \uparrow$$





One usually cannot control the given parameter σ. But one can achieve the same power of test by increasing the sample size *n*.

E.g. Test $H_0 : \mu = 100$ v.s. $H_1 : \mu > 100$ at $\alpha = 0.05$ with $\sigma = 14$ known.
Requirement: $1 - \beta = 0.60$ when $\mu = 103$.
Find smallest sample size *n*.

Remark: Two condisions: $\alpha = 0.05$ and $1 - \beta = 0.60$
Two unknowns: Critical value y^* and sample size *n*

Sol.

$$C = \left\{ z : z = \frac{\bar{y} - \mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \right\}.$$

$$
\begin{aligned}
1-\beta &= \mathbb{P}\left(\frac{\overline{Y}-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(\frac{\overline{Y}-\mu_1}{\sigma/\sqrt{n}} + \frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} \geq z_\alpha \,\middle|\, \mu_1\right) \\
&= \mathbb{P}\left(Z \geq -\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} + z_\alpha \,\middle|\, \mu_1\right) \\
&= \Phi\left(\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha\right)
\end{aligned}
$$

$$
\frac{\mu_1-\mu_0}{\sigma/\sqrt{n}} - z_\alpha = \Phi^{-1}(1-\beta) \quad \Longleftrightarrow \quad n = \left(\sigma \times \frac{\Phi^{-1}(1-\beta)+z_\alpha}{\mu_1-\mu_0}\right)^2
$$

$$
n = \left\lceil \left(14 \times \frac{0.2533+1.645}{103-100}\right)^2 \right\rceil = \lceil 78.48 \rceil = 79.
$$

□

R: $z_\alpha = \text{qnorm}(1-\alpha)$ and $\Phi^{-1}(1-\beta) = \text{qnorm}(1-\beta)$

Nonnormal data

Test $H_0 : \theta = \theta_0$, with $f_Y(y;\theta)$ is not normal distribution.

1. Identify a sufficient estimator $\widehat{\theta}$ for θ

2. Find the critical region C: Least compatible with H_0 but still admissible under H_1

3. Given $\alpha \rightarrow$ find $C \rightarrow \beta$, $1 - \beta$...

   From $C \rightarrow$ determine α

   From $\theta_e \rightarrow$ find P-value

Examples for nonnormal data

E.g. 1. A random sample of size n from uniform distr. $f_Y(y;\theta) = 1/\theta$, $y \in [0,\theta]$. To test

$$H_0 : \theta = 2.0 \quad \text{v.s.} \quad H_1 : \theta < 2.0$$

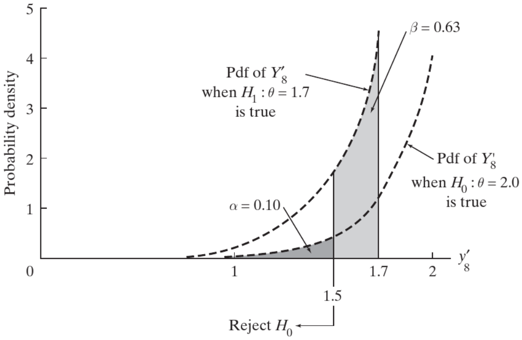
at the level $\alpha = 0.10$ of significance, one can use the decision rule based on Y_8'. Find the probability of committing a Type II error when $\theta = 1.7$.

Remark: Y_8' is a sufficient estimator for θ. Why?

Sol. 1) The critical region should has the form: $C = \{y_8' : y_8' \leq c\}$.

2) We need to use the condition $\alpha = 0.10$ to find c.

3) Find the prob. of Type II error.



$$f_{Y_{max}}(y) = ... = n\frac{y^{n-1}}{\theta^n} \quad y \in [0, \theta].$$

$$\alpha = \int_0^c n\frac{y^{n-1}}{\theta_0^n}\mathrm{d}y = \left(\frac{c}{\theta_0}\right)^n \quad \Longrightarrow \quad c = \theta_0\alpha^{1/n} \qquad \text{(Under } H_0 : \theta = \theta_0\text{)}$$

$$\beta = \int_{\theta_0\alpha^{1/n}}^1 n\frac{y^{n-1}}{\theta_1^n}\mathrm{d}y = 1 - \left(\frac{\theta_0}{\theta_1}\right)^n \alpha \qquad \text{(Under } \theta = \theta_1\text{)}$$

E.g. 2. A random sample of size 4 from Poisson(λ): $p_X(k;\lambda) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \cdots$. One wants to test

$$H_0 : \lambda = 0.8 \quad \text{v.s.} \quad H_1 : \lambda > 0.8.$$

at the level $\alpha = 0.10$. Find power of test when $\lambda = 1.2$.

Sol. 1) We've seen: $\overline{X} = \sum_{i=1}^{4} X_i$ is a sufficent estimator for λ;
$\overline{X}$ ~Poisson(3.2)

2) $C = \{\bar{k}; \bar{k} \geq c\}$.

3) $\alpha = 0.10 \to c = 6$.

4) Alternative $\lambda = 1.2 \to 1 - \beta = 0.35$.

Finding critical region

k	P(X=k)	P(X<= k)	P(X>k)	P(X>=k)
0	0.0408	0.0408	0.9592	1
1	0.1304	0.1712	0.8288	0.9592
2	0.2087	0.3799	0.6201	0.8288
3	0.2226	0.6025	0.3975	0.6201
4	0.1781	0.7806	0.2194	0.3975
5	0.114	0.8946	0.1054	0.2194
6	0.0608	0.9554	0.0446	0.1054
7	0.0278	0.9832	0.0168	0.0446
8	0.0111	0.9943	0.0057	0.0168
9	0.004	0.9982	0.0018	0.0057
10	0.0013	0.9995	0.0005	0.0018
11	0.0004	0.9999	0.0001	0.0005
12	0.0001	1	0	0.0001
13	0	1	0	0
14	0	1	0	0

Poisson lambda= 3.2

Computing power of test

k	P(X=k)	P(X<= k)	P(X>k)	P(X>=k)
0	0.0082	0.0082	0.9918	1
1	0.0395	0.0477	0.9523	0.9918
2	0.0948	0.1425	0.8575	0.9523
3	0.1517	0.2942	0.7058	0.8575
4	0.182	0.4763	0.5237	0.7058
5	0.1747	0.651	0.349	0.5237
6	0.1398	0.7908	0.2092	0.349
7	0.0959	0.8867	0.1133	0.2092
8	0.0575	0.9442	0.0558	0.1133
9	0.0307	0.9749	0.0251	0.0558
10	0.0147	0.9896	0.0104	0.0251
11	0.0064	0.996	0.004	0.0104
12	0.0026	0.9986	0.0014	0.004
13	0.0009	0.9995	0.0005	0.0014
14	0.0003	0.9999	0.0001	0.0005
15	0.0001	1	0	0.0001
16	0	1	0	0
17	0	1	0	0
18	0	1	0	0
19	0	1	0	0
20	0	1	0	0

Poisson lambda= 4.8

$$1 - \beta = \mathbb{P}\,(\text{Reject } H_0 \mid H_1 \text{ is true}) = \mathbb{P}(\overline{X} \geq 6 | \overline{X} \sim \textit{Poisson}(4.8))$$

```
PlotPoissonTable <- function(n=14,lambda=3.2,png_filename,TableTitle) {
   library(gridExtra)
   library(grid)
   library(gtable)
  x = seq(1,n,1)
  # qpois(0.90,lambda)
  tb = cbind(x,
             round(dpois(x,lambda),4),
             round(ppois(x,lambda),4),
             round(1-ppois(x,lambda),4),
             round(c(1,(1-ppois(x,lambda))[1:n]),4))
  colnames(tb) <- c("k", "P(X=k)", "P(X<= k)", "P(X>k)", "P(X>=k)")
  rownames(tb) <-x
  table <- tableGrob(tb,rows = NULL)
   title  <- textGrob(TableTitle,gp=gpar(fontsize=12))
  footnote <- textGrob(paste("Poisson lambda=",lambda),
                          x=0, hjust=0, gp=gpar( fontface="italic"))
  padding <- unit(0.2,"line")
  table <- gtable_add_rows(table, heights = grobHeight(title) + padding,pos = 0)
  table <- gtable_add_rows(table, heights = grobHeight(footnote)+ padding)
  table <- gtable_add_grob(table, list( title , footnote),
                              t=c(1, nrow(table)), l=c(1,2), r=ncol(table))
  png(png_filename)
  grid.draw(table)
  dev.off()
}

PlotPoissonTable(14,3.2,"Example_6-4-3_1.png","Finding critical region")
PlotPoissonTable(20,4.8,"Example_6-4-3_2.png","Computing power of test")
```

E.g. 3. A random sample of size 7 from $f_Y(y;\theta) = (\theta+1)y^\theta$, $y \in [0,1]$. Test

$$H_0 : \theta = 2.0 \quad \text{v.s.} \quad H_1 : \theta > 2.0$$

Decision rule: Let X be the number of y_i's that exceed 0.9 and reject H_0 if $X \geq 4$.
Find α.

Sol. 1) $X \sim \text{binomial}(7, p)$.
2) Find p:

$$\begin{aligned} p &= \mathbb{P}(Y \geq 0.9 | H_0 \text{ is true}) \\ &= \int_{0.9}^{1} 3y^2 \mathrm{d}y = 0.271 \end{aligned}$$

3) Compute α:

$$\alpha = \mathbb{P}(X \geq 4 | \theta = 2) = \sum_{k=4}^{7} \binom{7}{k} 0.271^k 0.729^{7-k} = 0.092.$$