Math 362: Mathmatical Statistics II

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Chapter 6. Hypothesis Testing

§ 6.5 A Notion of Optimality: The Generalized Likelihood Ratio

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Question:

- Vector parameter — Scale parameter
- Composite-vs-composite test — Simple-vs-composite test

$H_0 : \theta \in \omega$ vs $H_1 : \theta \in \Omega \cap \omega^c$ — $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

E.g. Two normal populations $N(\mu_i, \sigma_i)$, $i = 1, 2$. σ_i are known, μ_i unknown.

$$H_0 : \mu_1 = \mu_2 \quad \text{and} \quad H_1 : \mu_1 \neq \mu_2.$$

- Let $Y_1, \cdots, Y_n$ be a random sample of size n from $f_Y(y; \theta_1, \cdots, \theta_k)$
- Let Ω be all possible values of the parameter vector $(\theta_1, \cdots, \theta_k)$
- Let $\omega \subseteq \Omega$ be a subset of Ω.

- Test:

$$H_0 : \theta \in \omega \quad \text{vs} \quad H_1 : \theta \in \Omega \setminus \omega.$$

- The **generalized likelihood ratio, λ,** is defined as

$$\lambda := \frac{\max_{(\theta_1, \cdots, \theta_k) \in \omega} L(\theta_1, \cdots, \theta_k)}{\max_{(\theta_1, \cdots, \theta_k) \in \Omega} L(\theta_1, \cdots, \theta_k)}$$

- $\lambda \in (0, 1]$:
  λ close to one $\rightarrow$ data are compatible with H_0 $\rightarrow$ accept H_0
  λ close to zero $\rightarrow$ data are NOT compatible with H_0 $\rightarrow$ reject H_0

- **Generalized likelihood ratio test (GLRT)**: Use the following critical region

$$C = \{\lambda : \lambda \in (0, \lambda^*]\}$$

  to reject H_0 with either α or y^* being determined through

$$\alpha = \mathbb{P}\left(0 < \Lambda \leq \lambda^* \middle| H_0 \text{ is true}\right).$$

Remarks:

1. Maximization over Ω instead of $\Omega \setminus \omega$ in denominator:

   In practice, little effect on this change.

   In theory, much easier/nicer: $L(\theta_1, \cdots, \theta_k)$ is maximized over the whole space Ω by the max. likelihood estimates: $\Omega_e := (\theta_{e,1}, \cdots, \theta_{e,k}) \in \Omega$.

2. Suppose the maximization over ω is achieved at $\omega_e \in \omega$.

3. Hence:

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)}.$$

Remarks;

4. For simple-vs-composite test, $\omega = \{\omega_0\}$ consists only one point:

$$\lambda = \frac{L(\omega_0)}{L(\Omega_e)}.$$

5. Working with Λ is hard since $f_\Lambda(\lambda | H_0)$ is hard to obtain.

   If Λ is a *(monotonic) function* of some r.v. W, whose pdf is known.

   Suggesting testing procedure

$$\text{Test based on } \lambda \quad \Longleftrightarrow \quad \text{Test based on } w.$$

E.g. 1 Let $Y_1, \cdots, Y_n$ be a random sample of size n from the uniform pdf: $f_Y(y:\theta) = 1/\theta$, $y \in [0,\theta]$. Find the form of GLRT for

$$H_0 : \theta = \theta_0 \quad \text{v.s.} \quad H_1 : \theta < \theta_0 \qquad \text{with given } \alpha.$$

Sol. 1) Simple null hypothesis test $\Longrightarrow$

$$L(\omega_e) = L(\theta_0) = \theta_0^{-n} \prod_{i=1}^{n} I_{[0,\theta_0]}(y_i) = \theta^{-n} I_{[0,\theta_0]}(y_{max}).$$

2) MLE is $Y_{max} \Longrightarrow$

$$L(\Omega_e) = L(y_{max}) = y_{max}^{-n} I_{[0,y_{max}]}(y_{max}) = y_{max}^{-n}.$$

3) Hence,

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left(\frac{y_{max}}{\theta_0}\right)^n I_{[0,\theta_0]}(y_{max})$$

that is, the test statistic is

$$\Lambda = \left(\frac{Y_{max}}{\theta_0}\right)^n I_{[0,\theta_0]}(Y_{max}).$$

4) α and critical value λ^*:

$$\begin{aligned}\alpha &= \mathbb{P}(0 < \Lambda \leq \lambda^* | H_0 \text{ is true}) \\ &= \mathbb{P}\left(\left[\frac{Y_{max}}{\theta_0}\right]^n I_{[0,\theta_0]}(Y_{max}) \leq \lambda^* \middle| H_0 \text{ is true}\right) \\ &= \mathbb{P}\left(Y_{max} \leq \theta_0(\lambda^*)^{1/n} \middle| H_0 \text{ is true}\right)\end{aligned}$$

Λ suggests the test statistic Y_{max}:

Test based on $\lambda \Longleftrightarrow$ Test based of y_{max}

5) Let's find the pdf of Y_{max}. The cdf of Y is $F_Y(y;\theta_0) = y/\theta_0$ for $y \in [0, \theta_0]$. Hence,

$$
\begin{aligned}
f_{Y_{max}}(y;\theta_0) &= nF_Y(y;\theta_0)^{n-1} f_Y(y;\theta_0) \\
&= \frac{ny^{n-1}}{\theta_0^n}, \quad y \in [0, \theta_0].
\end{aligned}
$$

6) Finally, by setting $y^* := \theta_0(\lambda^*)^{1/n}$, we see that

$$
\begin{aligned}
\alpha &= \mathbb{P}\left(Y_{max} \le y^* \middle| H_0 \text{ is true}\right) \\
&= \int_0^{y^*} \frac{ny^{n-1}}{\theta_0^n} \mathrm{d}y \\
&= \frac{(y^*)^n}{\theta_0^n} \quad \Longleftrightarrow \quad y^* = \theta_0 \alpha^{1/n}.
\end{aligned}
$$

7) Therefore, H_0 is rejected if

$$
y_{max} \le \theta_0 \alpha^{1/n}.
$$

E.g. 2 Let $X_1, \cdots, X_n$ be a random sample from the geometric distribution with parameter p.

Find a test statistic Λ for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.

Sol. Since the null hypothesis is simple, we have that

$$L(\omega_e) = L(p_0) = \prod_{i=1}^{n} (1 - p_0)^{k_i - 1} p_0 = (1 - p_0)^{n\bar{k} - n} p_0^n,$$

which shows that $\bar{k}$ is a sufficient estimator.

On the other hand, the MLE for geometric distibution is $1/\bar{k}$. So

$$L(\Omega_e) = L(1/\bar{k}) = \prod_{i=1}^{n} \left(1 - \frac{1}{\bar{k}}\right)^{k_i - 1} \frac{1}{\bar{k}} = \left(\frac{\bar{k} - 1}{\bar{k}}\right)^{n\bar{k} - n} \frac{1}{\bar{k}^n}.$$

Hence,

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \left(\frac{\bar{k}(1 - p_0)}{\bar{k} - 1}\right)^{n\bar{k} - n} (p_0 \bar{k})^n$$

Finally, $\Lambda = \left(\frac{\overline{X}(1-p_0)}{\overline{X}-1}\right)^{n\overline{X}-n} (p_0 \overline{X})^n$. □

E.g. 3 Let $Y_1, \cdots, Y_n$ be a random sample from the exponential distribution with parameter λ.

Find a test statistic V for testing $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

Sol. Since the null hypothesis is simple,

$$L(\omega_e) = L(\lambda_0) = \prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 y_i} = \lambda_0^n e^{-\lambda_0 \sum_{i=1}^{n} y_i}$$

Let $Z = \sum_{i=1}^{n} Y_i \sim \text{Gamma}(n, \lambda)$, which is a sufficient estimator.

On the other hand, the MLE for λ is $1/\bar{y} = n/z$:

$$L(\Omega_e) = L(1/\bar{y}) = (n/z)^n e^{-n}.$$

Hence,

$$v = \frac{L(\omega_e)}{L(\Omega_e)} = z^n n^{-n} \lambda_0^n e^{-\lambda_0 z + n}$$

Finally, $V = Z^n n^{-n} \lambda_0^n e^{-\lambda_0 Z + n}$ or $V = Z^n e^{-\lambda_0 Z}$ ... $\square$

E.g. 4 Let $Y_1, \cdots, Y_n$ be a random sample from $N(\mu, 1)$.
Find a test statistic Λ for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$.

Sol. Since the null hypothesis is simple,

$$L(\omega_e) = L(\mu_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\mu_0)^2}{2}}.$$

On the other hand, the MLE for μ is $\bar{y}$:

$$L(\Omega_e) = L(\bar{y}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_i-\bar{y})^2}{2}}.$$

Hence,

$$\lambda = \frac{L(\omega_e)}{L(\Omega_e)} = \exp\left(-\sum_{i=1}^{n} \frac{(y_i-\mu_0)^2 - (y_i-\bar{y})^2}{2}\right) = \exp\left(-\frac{n(\bar{y}-\mu_0)^2}{2}\right).$$

Finally, $\Lambda = \exp\left(-\frac{n}{2}\left(\overline{Y}-\mu_0\right)^2\right)$ or $V = \frac{\overline{Y}-\mu_0}{1/\sqrt{n}} \sim N(0,1)$ □