Homework 1 Due on Jan. 21

§5.2: 1, 2, 3, 4, 5, 6, 7, 12, 17, 18, 19, 20, 21, 23.
§5.4: 4, 6, 7, 9, 11, 14, 17, 19.

§5.2: Estimating Parameters: MLE and MME

Question 5.2.1 A random sample of size 8:

$$X_1 = 1,\ X_2 = 0,\ X_3 = 1,\ X_4 = 1,\ X_5 = 0,\ X_6 = 1,\ X_7 = 1,\ X_8 = 0,$$

is taken from the probability function

$$p_X(k;\theta) = \theta^k(1-\theta)^{1-k}, \quad k = 0, 1; \quad 0 < \theta < 1.$$

Find the maximum likelihood estimate for θ

Question 5.2.2 The number of red chips and white chips in an urn is unknown, but the proportion, p, of reds is either $1/3$ or $1/2$. A sample of size 5, drawn with replacement, yields the sequence red, white, white, red, and white. What is the maximum likelihood estimate for p?

Question 5.2.3 Use the sample

$$Y_1 = 8.2,\ Y_2 = 9.1,\ Y_3 = 10.6,\ Y_4 = 4.9$$

to calculate the maximum likelihood estimate for λ in the exponential pdf

$$f_Y(y;\lambda) = \lambda e^{-\lambda y}, \quad y \geqslant 0.$$

Question 5.2.4 Suppose a random sample of size n is drawn from the probability model

$$p_X(k;\theta) = \frac{\theta^{2k} e^{-\theta^2}}{k!}, \quad k = 0, 1, 2, \cdots$$

Find a formula for the maximum likelihood estimator $\widehat{\theta}$.

Question 5.2.5 Given that

$$Y_1 = 2.3,\ Y_2 = 1.9,\ Y_3 = 4.6$$

is a random sample from

$$f_Y(y;\theta) = \frac{y^3 e^{-y/\theta}}{6\theta^4}, \quad y \geqslant 0$$

calculate the maximum likelihood estimate for θ.

Question 5.2.6 Use the method of maximum likelihood to estimate θ in the pdf

$$f_Y(y;\theta) = \frac{\theta}{2\sqrt{y}} e^{-\theta\sqrt{y}}, \quad y \geqslant 0$$

Evaluate θ_e for the following random sample of size 4

$$Y_1 = 6.2,\ Y_2 = 7.0,\ Y_3 = 2.5,\ Y_4 = 4.2.$$

Question 5.2.7 An engineer is creating a project scheduling program and recognizes that the tasks making up the project are not always completed on time. However, the completion proportion tends to be fairly high. To reflect this condition, he uses the pdf

$$f_Y(y;\theta) = \theta y^{\theta-1}, \quad y \in [0,1], \quad \theta > 0$$

where y is the proportion of the task completed. Suppose that in his previous project, the proportions of tasks completed were $0.77, 0.82, 0.92, 0.94$, and 0.98. Estimate θ.

Question 5.2.12 A random sample of size n is taken from the pdf

$$f_Y(y;\theta) = \frac{2y}{\theta^2}, \quad y \in [0,\theta].$$

Find an expression for $\widehat{\theta}$, the maximum likelihood estimator for θ.

Question 5.2.17 Let $y_1, y_2, \cdots, y_n$ be a random sample of size n from the pdf

$$f_Y(y;\theta) = \frac{2y}{\theta^2}, \quad y \in [0,\theta].$$

Find a formula for the method of moments estimate for θ. Compare the values of the method of moments estimate and the maximum likelihood estimate if a random sample of size 5 consists of the numbers $17, 92, 46, 39$, and 56.

Question 5.2.18 Use the method of moments to estimate θ in the pdf

$$f_Y(y;\theta) = (\theta^2 + \theta) y^{\theta-1}(1-y), \quad y \in [0,1].$$

Assume that a random sample of size n has been collected.

Question 5.2.19 A criminologist is searching through FBI files to document the prevalence of a rare double-whorl fingerprint. Among six consecutive sets of 100,000 prints scanned by a computer, the numbers of persons having the abnormality are 3, 0, 3, 4, 2, and 1, respectively. Assume that double whorls are Poisson events. Use the method of moments to estimate their occurrence rate, λ. How would your answer change if λ were estimated using the method of maximum likelihood?

Question 5.2.20 Find the method of moments estimate for λ if a random sample of size n is taken from the exponential pdf,

$$f_Y(y;\lambda) = \lambda e^{-\lambda y}, \quad y \geqslant 0.$$

Question 5.2.21 Suppose that

$$Y_1 = 8.3,\ Y_2 = 4.9,\ Y_3 = 2.6,\ Y_4 = 6.5$$

is a random sample of size 4 from the two-parameter uniform pdf,

$$f_Y(y;\theta_1,\theta_2) = \frac{1}{2\theta_2}, \quad \theta_1 - \theta_2 \leqslant y \leqslant \theta_1 + \theta_2.$$

Use the method of moments to calculate θ_{1e} and θ_{2e}.

Question 5.2.23 Calculate the method of moments estimate for the parameter θ in the probability function

$$p_X(k;\theta) = \theta^k(1-\theta)^{1-k}, \quad k = 0, 1$$

if a sample of size 5 is the set of numbers 0, 0, 1, 0, 1.

Question 5.4.4 A sample of size $n = 16$ is drawn from a normal distribution where $\sigma = 10$ but μ is unknown. If $\mu = 20$, what is the probability that the estimator $\widehat{\mu} = \overline{Y}$ will lie between 19.0 and 21.0?

Question 5.4.6 Let Y_{min} be the smallest order statistic in a random sample of size n drawn from the uniform pdf, $f_Y(y;\theta) = 1/\theta$, $0 \leqslant y \leqslant \theta$. Find an unbiased estimator for θ based on Y_{min}.

Question 5.4.7 Let Y be the random variable with pdf

$$f_Y(y,\theta) = e^{-(y-\theta)}, \quad y \geqslant \theta, \quad \theta > 0$$

Show that $Y_{min} - \frac{1}{n}$ is an unbiased estimator of θ.

Question 5.4.9 A random sample of size 2, Y_1 and Y_2 , is drawn from the pdf

$$f_Y(y;\theta) = 2y\theta^2, \quad 0 < y < 1/\theta.$$

What must c equal if the statistic $c(Y_1 + 2Y_2)$ is to be an unbiased estimator for $1/\theta$?

Question 5.4.11 Suppose that W is an unbiased estimator for θ. Can W^2 be an unbiased estimator for θ^2?

Question 5.4.14 Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n from the pdf

$$f_Y(y;\theta) = \frac{1}{\theta}e^{-y/\theta}., \quad y > 0.$$

1. Let $\widehat{\theta} = nY_{min}$. Is $\widehat{\theta}$ unbiased for θ?
2. Let $\widehat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. Is $\widehat{\theta}$ unbiased for θ?

Question 5.4.17 Let $X_1, X_2, \cdots, X_n$ denote the outcomes of a series of n independent trials, where

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

for $i = 1, \cdots, n$. Let $X = X_1 + X_2 + \cdots + X_n$.

1. Show that $\widehat{p}_1 = X_1$ and $\widehat{p}_2 = \frac{X}{n}$ are unbiased estimators for p.
2. Intuitively, $\widehat{p}_2$ is a better estimator than $\widehat{p}_1$ because $\widehat{p}_1$ fails to include any of the information about the parameter contained in trials 2 through n. Verify that speculation by comparing the variances of $\widehat{p}_1$ and $\widehat{p}_2$.

Question 5.4.19 Let $Y_1, Y_2, \cdots, Y_na$ be a random sample of size n from the pdf

$$f_Y(y;\theta) = \frac{1}{\theta}e^{-y/\theta}, \quad y > 0.$$

1. Show that $\widehat{\theta}_1 = Y_1$, $\widehat{\theta}_2 = \overline{Y}$ and $\widehat{\theta}_3 = nY_{min}$ are all unbiased estimators for θ.
2. Find the variances of $\widehat{\theta}_i$, $i = 1, 2, 3$.
3. Calculate the relative efficiencies of $\widehat{\theta}_1$ to $\widehat{\theta}_3$ and $\widehat{\theta}_2$ to $\widehat{\theta}_3$.