Emory - Math362 | Probability & Statistics II
Instructor: Le Chen | 2020 Spring

Homework 3 | Due on Feb. 4

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§5.6: 2, 4, 7, 9, 11.
§5.7: 2, 3.
§5.8: 1, 2, 4, 6, 7.

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§5.6: Sufficient Estimators

Question 5.6.2 Let X_1, X_2, and X_3 be a set of three independent Bernoulli random variables with unknown parameter $p = P(X_i = 1)$. It was shown in the class that $\widehat{p} = X_1 + X_2 + X_3$ is sufficient for p. Show that the linear combination $\widehat{p}^* = X_1 + 2X_2 + 3X_3$ is not sufficient for p.

Question 5.6.4 Show that $\widehat{\sigma}^2 = \sum_{i=1}^n Y_i^2$ is sufficient of σ^2 if $Y_1, \cdots, Y_n$ is a random sample from a normal pdf with $\mu = 0$.

Question 5.6.7 Suppose a random sample of size n is drawn from the pdf

$$f_Y(y; \theta) = e^{-(y-\theta)}, \quad y \geqslant \theta.$$

(a) Show that $\widehat{\theta} = Y_{min}$ is sufficient for the threshold parameter θ.

(b) Show that Y_{max} is not sufficent for θ.

Question 5.6.9 A pdf $g_W(w; \theta)$ is said to be expressed in exponential form if it can be written as

$$g_W(w; \theta) = e^{K(w)p(\theta)+S(w)+q(\theta)}$$

where the range of W is independent of θ. Show that $\widehat{\theta} = \sum_{i=1}^n K(W_i)$ is sufficient for θ.

Question 5.6.11 Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a Pareto pdf,

$$f_Y(y; \theta) = \frac{\theta}{(1+y)^{1+\theta}}, \quad y \geqslant 0, \quad \theta > 0.$$

Write $f_Y(y; \theta)$ in exponential form and deduce a sufficient statistic for θ.

Question 5.7.2 Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a normal pdf having $\mu = 0$. Show that $S_n^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator for $\sigma^2 = \text{Var}(Y)$.

Question 5.7.3 Suppose $Y_1, Y_2, \cdots, Y_n$ is a random sample from the exponential pdf, $f_Y(y; \lambda) = \lambda e^{-\lambda y}$, $y > 0$.

(a) Show that $\widehat{\lambda}_n = Y_1$ is not consistent for λ.

(b) Show that $\widehat{\lambda}_n = \sum_{i=1}^{n} Y_i$ is not consistent for λ.

Question 5.8.1 Suppose that X is a geometric random variable, where $p_X(k|\theta) = (1 - \theta)^{k-1}\theta$, $k = 1, 2, \cdots$. Assume that the prior distribution for θ is the beta pdf with parameters r and s. Find the posterior distribution for θ.

Question 5.8.2 Suupose that X follow the binomial distribution with a known parameter n and an unknown parameter θ. Find the square-error loss Bayes estimate of θ if θ follows the Beta distribution with parameters r and s. Express this estimate as a weighted avereage of the maximum likelihood estimate for θ and the mean of the prior pdf.

Question 5.8.4 Suupose that X follow the binomial distribution with a known parameter n and an unknown parameter θ. What is the squared-error loss Bayes estimate for the parameter θ in a binomial pdf, where θ has a uniform distribution—that is, a noninformative prior? (Recall that a uniform prior is a beta pdf for which $r = s = 1$.)

Question 5.8.6 Suppose that Y is a gamma random variable with parameters r and θ and the prior is also gamma with parameters s and μ. Here r and s are called *shape parameters* and β and μ *rate parameters*:

$$f_Y(y; \theta) = \frac{\theta^r}{\Gamma(r)} y^{r-1} e^{-\theta y}, \qquad y \geqslant 0.$$

Assume that the rate parameter θ is unknown and the shape parameter r is known. Show that the posterior pdf is gamma with parameters $r + s$ and $y + \mu$.

Question 5.8.7 Let $Y_1, Y_2, \cdots, Y_n$ be a random sample from a gamma pdf with a known shape parameter r and an unknown rate parameter θ , where the prior distribution assigned to θ is the gamma pdf with parameters s and μ. Let $W = Y_1 + Y_2 + \cdots + Y_n$. Find the posterior pdf for θ.