Homework 5 Due on Feb. 25

§6.5: 1, 2, 3, 5.
§7.3: 2, 4, 7, 8, 9, 11.

§6.5: A Notion of Optimality: The Generalized Likelihood Ratio

Question 6.5.1 Let $k_1, k_2, \cdots, k_n$ be a random sample from the geometric probability function

$$p_X(k; p) = (1-p)^{k-1} p, \quad k = 1, 2, \cdots$$

Find λ, the generalized likelihood ratio for testing $H_0 : p = p_0$ versus $H_1 : p \neq p_0$.
Hint: The maximum likelihood estimate for p is $1/\bar{k}$.

Question 6.5.2 Let $y_1, y_2, ..., y_{10}$ be a random sample from an exponential pdf with unknown parameter λ: $f_Y(y : \lambda) = \lambda e^{-\lambda y}$ for $y \geqslant 0$. Find the form of the GLRT for $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. What integral would have to be evaluated to determine the critical value if α were equal to 0.05?
Hint: The maximum likelihood estimate for λ is $1/\bar{y} = n/y$ with $y := \sum_{i=1}^n y_i$.

Question 6.5.3 Let $y_1, y_2, \cdots, y_n$ be a random sample from a normal pdf with unknown mean μ and variance 1. Find the form of the GLRT for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ (that is, find the form of the critical region.)
Hint: The maximum likelihood estimate for μ is $\bar{y}$.

Question 6.5.5 Let k denote the number of successes observed in a sequence of n independent Bernoulli trials, where $p = \mathbb{P}(\text{success})$.
(a) Show that the critical region of the likelihood ratio test of $H_0 : p = 1/2$ versus $H_1 : p = 1/2$ can be written in the form

$$k \ln(k) + (n-k) \ln(n-k) \geqslant \lambda^{**}$$

(b) Show that the critical region can be written in the form

$$\left| \bar{k} - \frac{1}{2} \right| \geqslant c$$

where c is determined by α and $\bar{k} = k/n$.
(Hints: use the symmetry of the function $f(t) = t \ln(t) + (1-t) \ln(1-t)$ for $t \in [0, 1]$.)

Question 7.3.2 Find the moment-generating function for a chi square random variable and use it to show that $\mathbb{E}(\chi_n^2) = n$ and $\text{Var}(\chi_n^2) = 2n$.

Hint: Moment generating function for the gamma distribution with the shape parameter r and the rate parameter λ is equal to

$$M(t) = (1 - t/\lambda)^{-r}, \quad \forall t < \lambda.$$

Question 7.3.4 Use the fact that $(n-1)S^2/\sigma^2$ is a chi square random variable with $n-1$ degree of freedom to prove that

$$\text{Var}(S^2) = \frac{2\sigma^4}{n-1}.$$

Hint: Use the fact that the variance of a chi square random variable with k df is $2k$.

Question 7.3.7 Use Appendix Table A.4 to find
(a) $F_{0.50,6,7}$
(b) $F_{0.001,15,5}$
(c) $F_{0.90,2,2}$

Question 7.3.8 Let V and U be independent chi square random variables with 7 and 9 degrees of freedom, respectively. Is it more likely that $\frac{V/7}{U/9}$ will be between (1) 2.51 and 3.29 or (2) 3.29 and 4.20?

Question 7.3.9 Use Appendix Table A.4 to find the values of x that
(a) $\mathbb{P}(0.109 < F_{4,6} < x) = 0.95$.
(b) $\mathbb{P}(0.427 < F_{11,7} < 1.69) = x$.
(c) $\mathbb{P}(F_{x,x} > 5.35) = 0.01$.
(d) $\mathbb{P}(0.115 < F_{3,x} < 3.29) = 0.90$.
(e) $\mathbb{P}\left(x < \frac{V/2}{U/3}\right) = 0.25$ where $V \sim$ Chi Square(2), $U \sim$ Chi Square(3), and $U \perp V$.

Question 7.3.11 If the random variable F has an F distribution with m and n degrees of freedom, show that $1/F$ has an F distribution with n and m degrees of freedom.