Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 1. Probability

§ 1.1 Properties of Probability

§ 1.2 Methods of Enumerations

§ 1.3 Conditional Probability

§ 1.4 Independent Events

§ 1.5 Bayes Theorem

Sample space

Definition 1.1-1 Experiments for which the outcome cannot be predicted with certainty are called *random experiments*.

Definition 1.1-2 The collection of all possible outcomes is called the *sample space*, denoted as S.

Example 1.1-1 Find the sample space for the experiment of tossing a coin for (a) once (b) twice.

Example 1.1-2 Find the sample space for the experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears.

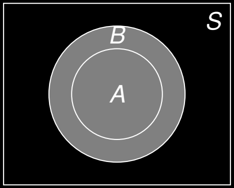
Example 1.1-3 Find the sample space for the experiment of measuring (in hours) the lifetime of a transistor.

Event

Definition 1.1-3 An *event*, denoted as A, is a collection of outcomes in S, that is, $A \subset S$. In particular, $S \subset S$.

Definition 1.1-4 When a random experiment is performed and an outcome of the experiment is in A, we say that *event A has occurred*.

$A \subset B$



Example 1.1-4 Consider the experiments in Examples 1.1-1 and 1.1-2.
(a) Let A be the event that the number of tosses required until the first head appears is even.

Example 1.1-4 Consider the experiments in Examples 1.1-1 and 1.1-2.
(b) Let B be the event that the number of tosses required until the first head appears is odd.

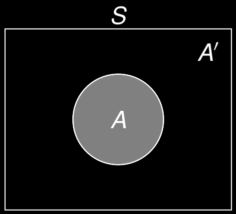
Example 1.1-4 Consider the experiments in Examples 1.1-1 and 1.1-2.
(c) Let C be the event that the number of tosses required until the first head appears is less than 6.

Algebra of sets

Definition 1.1-5 (Equality) Two sets A and B are *equal*, denoted $A = B$, if and only if $A \subset B$ and $B \subset A$.

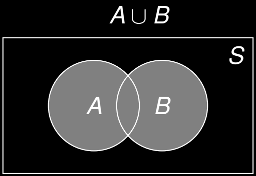
Definition 1.1-6 (Complementation) Suppose $A \subset S$. The *complement* of set A, denoted A' or $\overline{A}$, is the set containing all elements in S but not in A, namely,

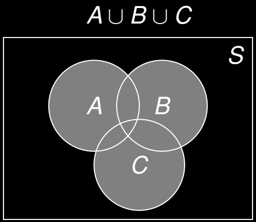
$$A' := \{x : x \in S \text{ and } x \notin A\} .$$



Definition 1.1-7 (Union) The *union* of sets A and B, denoted $A \cup B$, is the set containing all elements in either A or B or both, i.e.,

$$A \cup B := \{x : x \in A \text{ or } x \in B\} .$$

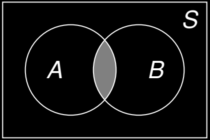




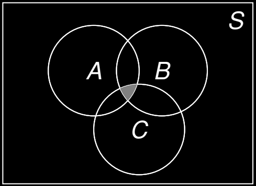
Definition 1.1-8 (Intersection) The *intersection* of sets A and B, denoted $A \cap B$, is the set containing all elements in both A and B.

$$A \cap B = \{x : x \in A \text{ and } B\}$$

$A \cap B$



$A \cap B \cap C$

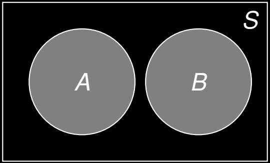


Definition 1.1-9 (Null set) The set containing no element is called the *null set,* denoted $\emptyset$. Note that

$$\emptyset = S'.$$

Definition 1.1-10 (Disjoit sets) Two sets A and B are called *disjoint* or *mutually exclusive* if they contain no common element, that is, if $A \cap B = \emptyset$.

A and B are disjoint



The definitions of the union and intersection of two sets can be extended to any finite number of sets as follows:

$$\begin{aligned}\bigcup_{i=1}^{n} A_i &= A_1 \cup A_2 \cup \cdots \cup A_n \\ &= \{x : x \in A_1 \text{ or } A_2 \text{ or } \cdots \text{ or } x \in A_n\}, \\ \bigcap_{i=1}^{n} A_i &= A_1 \cap A_2 \cap \cdots \cap A_n \\ &= \{x : x \in A_1 \text{ and } A_2 \text{ and } \cdots \text{ and } x \in A_n\},\end{aligned}$$

and even to countably infinite many sets:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Interpretation of sets in probability

1. $$S = \text{the certain event}$$
$$\emptyset = \text{the impossible event}$$

2. If A and B are events in S, then

$$A' = \text{the event that } A \text{ did not occur}$$
$$A \cup B = \text{the event that either } A \text{ or } B \text{ or both occurred}$$
$$A \cap B = \text{the event that both } A \text{ and } B \textit{ occurred}$$

3. Similarly, if $A_1, A_2, ..., A_n$ are sequence of events in S, then

$$\bigcup_{i=1}^{n} A_i = \text{the event that at least one of the } A_i \text{ occurred}$$

$$\bigcap_{i=1}^{n} A_i = \text{the event that all of the } A_i \text{ occurred}$$

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Definition 1.1-11 (Mutually exclussive and exhaustive events) Events $A_1, A_2, ..., A_k$ are *mutually exclusive and exhaustive* if

1. $A_i \cap A_j = \emptyset, i \neq j$, and (mutually exclusive)
2. $A_1 \cup A_2 \cup \cdots \cup A_k = S$. (exhaustive)

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Identities

By the above set of definitions, we obtain the following identities:

$$S' = \emptyset$$

$$A'' = A$$

$$S \cup A = S$$

$$S \cap A = A$$

$$A \cup A' = S$$

$$A \cap A' = \emptyset$$

Commutative Laws:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Laws:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morrgan's Laws:

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

The distributive laws can be extended as follows:

$$A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

$$A \cup \left(\bigcap_{i=1}^{n} B_i \right) = \bigcap_{i=1}^{n} (A \cup B_i)$$

Similarly, De Morgan's laws also can be extended as follows:

$$\left(\bigcup_{i=1}^{n} A_i \right)' = \bigcap_{i=1}^{n} A_i{}'$$

$$\left(\bigcap_{i=1}^{n} A_i \right)' = \bigcup_{i=1}^{n} A_i{}'$$

Example 1.1-5 An experiment consists of tossing two dice.
(a) Find the sample space S.

Example 1.1-5 An experiment consists of tossing two dice.
(b) Find the event A that the sum of the dots on the dice equals 7.

Example 1.1-5 An experiment consists of tossing two dice.
(c) Find the event B that the sum of the dots on the dice is greater than 10.

Example 1.1-5 An experiment consists of tossing two dice.
(d) Find the event C that the sum of the dots on the dice is greater than 12.

Example 1.1-6 Consider the experiment of Example 1.1-2. We define the events

$$A = \{k : k \text{ is odd}\}$$
$$B = \{k : 4 \leq k \leq 7\}$$
$$C = \{k : 1 \leq k \leq 10\}$$

where k is the number of tosses required until the first H (head) appears. Determine the events A', B', C', $A \cup B$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$, and $A' \cap B$.

Properties of Probability

Definition 1.1-12 (Relative frequency definition) Suppose that the random experiment is repeated n times. If event A occurs $\mathcal{N}(A)$ times, then the probability of event A, denoted by $\mathbb{P}(A)$, is defined as

$$\mathbb{P}(A) = \lim_{n \to \infty} \frac{\mathcal{N}(A)}{n},$$

where $\mathcal{N}(A)/n$ is relative frequency of an event A.

Remark 1.1-1 Note that this limit may not exist.

It is clear that for any event A, the relative frequency of A has the following properties:

1. $0 \leq \mathcal{N}(A)/n \leq 1$, where $\mathcal{N}(A)/n = 0$ if A occurs in none of the repeated trials and $\mathcal{N}(A)/n = 1$ if A occurs in all of the n repeated trials.

2. If A and B are mutually exclusive events, then

$$\mathcal{N}(A \cup B) = \mathcal{N}(A) + \mathcal{N}(B)$$

and

$$\frac{\mathcal{N}(A \cup B)}{n} = \frac{\mathcal{N}(A)}{n} + \frac{\mathcal{N}(B)}{n}.$$

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Example 1.1-7 A fair six-sided die is rolled six times. If the face numbered k is the outcome on roll k for $k = 1, 2, \cdots, 6$, we say that a match has occurred. The experiment is called a *success* if at least one match occurs during the six trials. Otherwise, the experiment is called a *failure*. The sample space is

$$S = \{\text{success}, \ \text{failure}\}.$$

Let $A = \{\text{success}\}$. We would like to assign a value to $\mathbb{P}(A)$. Accordingly, this experiment was simulated on a computer as follows:

No. of trials n	$\mathcal{N}(A)$	$\mathcal{N}(A)/n$
10	7	0.700
100	69	0.690
500	336	0.672
2000	1307	0.653
46656	31042	0.665

The probability of event A is not intuitively obvious, but it will be shown latter that

$$\mathbb{P}(A) = 1 - \left(1 - \frac{1}{6}\right)^6 = \frac{31031}{46656} \approx 0.665.$$

This assignment is certainly supported by the simulation (although not proved by it).

```python
#!/usr/bin/env python3

import random
random.seed(10)

def Experiment(trials):
    success = 0
    for k in range(0, trials):
        for i in range(1, 7):
            n = random.randint(1, 6)
            if n == i:
                success = success + 1
                break
    print("Number and ratio of success among {} tries are {} and {:0.3}, respectively".format(trials, success, success/trials))


def main():
    Experiment(10)
    Experiment(100)
    Experiment(500)
    Experiment(2000)
    Experiment(46656)


if __name__ == "__main__":
    main()
```

```
Number and ratio of success among 10 tries are 7 and 0.7, respectively
Number and ratio of success among 100 tries are 69 and 0.69, respectively
Number and ratio of success among 500 tries are 336 and 0.672, respectively
Number and ratio of success among 2000 tries are 1307 and 0.653, respectively
Number and ratio of success among 46656 tries are 31042 and 0.665, respectively
```

Axiomatic definition

Definition 1.1-13 (Probability) *Probability* is a real-valued set function $\mathbb{P}$ that assigns, to each event A in the sample space S, a number $\mathbb{P}(A)$, called the *probability* of the event A, such that the following properties are satisfied:

(a) $P(A) \geq 0$;

(b) $P(S) = 1$;

(c) If $A_1, A_2, A_3, \cdots$ are events and $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_k)$$

for each positive integer K, and

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

for an infinite, but countable, number of events.

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for an infinite, but countable, number of events.

Example 1.1-8 (Rolling a die once) Consider rolling a die once. For each subset A of $S = \{1, 2, 3, 4, 5, 6\}$, let $\mathbb{P}(A)$ be the number of elements of A divided by 6. It is trivial to see that this satisfies the first two axioms. There are only finitely many distinct collections of nonempty disjoint events. It is not difficult to see that Axiom (c) is also satisfied by this example.

Example 1.1-9 (Loaded die) If we believe that the die is loaded, we might believe that some sides have different probabilities of turning up. To be specific, suppose that we believe that 6 is three times as likely to come up as each of the other five sides. We could set $p_i = 1/8$ for $i = 1, 2, 3, 4, 5$ and $p_6 = 3/8$. Then, for each event A, define $\mathbb{P}(A)$ to be the sum of all p_i such that $i \in A$. For example, if $A = \{1, 4, 5\}$, then

$$\mathbb{P}(A) = p_1 + p_4 + p_5 = \frac{3}{8}.$$

It is not difficult to check that this also satisfies all three axioms.

Theorem 1.1-1 For each event A, we have $\mathbb{P}(A) = 1 - \mathbb{P}(A')$.

Theorem 1.1-2 $\mathbb{P}(\emptyset) = 0.$

Theorem 1.1-3 If events A and B are such that $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Theorem 1.1-4 For each event A, it holds that $0 \leq \mathbb{P}(A) \leq 1$.

Theorem 1.1-5 If A and B are any two events, then

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)\,.$$

Theorem 1.1-6 If A, B, and C are any three events, then

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = & \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ & - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \\ & + \mathbb{P}(A \cap B \cap C).\end{aligned}$$

Definition 1.1-14 Consider a finite sample space S with m elements

$$S = \{e_1, e_2, ..., e_m\},$$

where e_i is a possible outcomes of the experiment. If each of these outcomes has the same probability of occurring, we say that the m outcomes are *equally likely*, that is,

$$\mathbb{P}(e_i) = \frac{1}{m}, i = 1, 2, ..., m,$$

and

$$\mathbb{P}(A) = \frac{\mathcal{N}(A)}{m},$$

where $\mathcal{N}(A)$ is the number of outcomes belonging to event A and $m = \mathcal{N}(S)$ is the number of sample points in S.

Definition 1.1-15 Let $\mathbb{P}(A) = 0.9, \mathbb{P}(B) = 0.8$. Show that $\mathbb{P}(A \cap B) \geq 0.7$.

Example 1.1-10 Given that $\mathbb{P}(A) = 0.9, \mathbb{P}(B) = 0.8$, and $\mathbb{P}(A \cap B) = 0.75$, find
(a) $\mathbb{P}(A \cup B)$;

Example 1.1-10 Given that $\mathbb{P}(A) = 0.9, \mathbb{P}(B) = 0.8$, and $\mathbb{P}(A \cap B) = 0.75$, find (b) $\mathbb{P}(A \cap B')$;

Example 1.1-10 Given that $\mathbb{P}(A) = 0.9, \mathbb{P}(B) = 0.8$, and $\mathbb{P}(A \cap B) = 0.75$, find (c) $\mathbb{P}(A' \cap B')$.

Example 1.1-11 Let A, B, and C be three events in S. If

$$\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{4}, \quad \mathbb{P}(C) = \frac{1}{3}, \quad \mathbb{P}(A \cap B) = \frac{1}{8}, \quad \mathbb{P}(A \cap C) = \frac{1}{6}$$

and $\mathbb{P}(B \cap C) = 0$, find $\mathbb{P}(A \cup B \cup C)$.

Example 1.1-12 Prove that

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i) \quad \text{and} \quad \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} \mathbb{P}(A'),$$

where the second inequality above is known as the *Bonferroni inequality*.

Example 1.1-13 During a visit to a primary care physician's office, the probability of having neither lab work nor referral to a specialist is 0.21. Of those coming to that office, the probability of having lab work is 0.41 and the probability of having referral is 0.53. What is the probability of having both lab work and a referral?

Example 1.1-14 Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns $1/52$ to each of the 52 outcomes. Let

$$A = \{x : x \text{ is a jack, queen, or king}\},$$
$$B = \{x : x \text{ is a } 9, 10, \text{ or jack and } x \text{ is red}\},$$
$$C = \{x : x \text{ is a club}\},$$
$$D = \{x : x \text{ is a diamond, a heart, or a spade}\}.$$

Find (a) $\mathbb{P}(A)$;

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Find (b) $\mathbb{P}(A \cap B)$;

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Find (c) $\mathbb{P}(A \cup B)$;

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Find (d) $\mathbb{P}(C \cup D)$;

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\end{aligned}
$$

Find (e) $\mathbb{P}(C \cap D)$.

Example 1.1-15 Consider two events A and B such that $\mathbb{P}(A) = 1/3$ and $\mathbb{P}(B) = 1/2$. Determine the value of $\mathbb{P}(B \cap A')$ for each of the following conditions: (a) A and B are disjoint; (b) $A \subset B$; (c) $\mathbb{P}(A \cap B) = 1/8$.

Exercises from textbook: Section 1.1: 1, 2, 4, 5, 6, 7, 8, 9, 12, 13.