Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 1. Probability

A major use of probability in statistical inference is the updating of probabilities when certain events are observed.

The updated probability of event A after we learn that event B has occurred is the conditional probability of A given B.

Example 1.3-1 (Lottery) Consider a state lottery game in which six numbers are drawn without replacement from a bin containing the numbers $1 - 30$.

Each player tries to match the set of six numbers that will be drawn without regard to the order in which the numbers are drawn.

Suppose that you hold a ticket in such a lottery with the numbers

$$1, 14, 15, 20, 23, \text{ and } 27.$$

You turn on your television to watch the drawing but all you see is one number, 15, being drawn when the power suddenly goes off in your house. You don't even know whether 15 was the first, last, or some in-between draw.

However, now that you know that 15 appears in the winning draw, the probability that your ticket is a winner must be higher than it was before you saw the draw. How do you calculate the revised probability?

Example 1.3-1 is typical of the following situation. An experiment is performed for which the sample space S is given (or can be constructed easily) and the probabilities are available for all of the events of interest.

We then learn that some event B has occurred, and we want to know how the probability of another event A changes after we learn that B has occurred.

In Example 1.3-1, the event that we have learned is

$$B = \{\text{one of the numbers drawn is } 15\}.$$

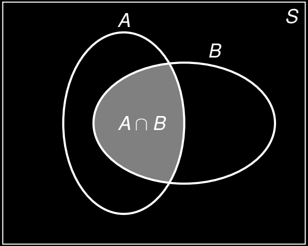
We are certainly interested in the probability of

$$A = \{\text{the numbers } 1, 14, 15, 20, 23, \text{ and } 27 \text{ are drawn}\},$$

and possibly other events.

If we know that the event B has occurred, then we know that the outcome of the experiment is one of those included in B. Hence, to evaluate the probability that A will occur, we must consider the set of those outcomes in B that also result in the occurrence of A.

As sketched in the figure, this set is precisely the set $A \cap B$. It is therefore natural to calculate the revised probability of A according to the following definition.



Definition 1.3-1 (Conditional probability) *Conditional probability of event A given event B*, is defined by

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \qquad \text{provided that } \mathbb{P}(B) > 0.$$

Similarly,

$$\mathbb{P}(B|A) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}, \qquad \text{provided } \mathbb{P}(A) > 0.$$

Using the above two equations, we have the following multiplication rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)P(B) = \mathbb{P}(B|A)P(A).$$

This often quite useful in computing the joint probability of events A and B.

In general, suppose that $A_1, A_2, \cdots, A_n$ are events such that $\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_{n-1}) > 0$. Then

$$\begin{gathered} P(A_1 \cap A_2 \cap \cdots \cap A_n) \\ \| \\ P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \end{gathered}$$

Solution (Example 1.3-1). In Example 1.3-1, you learned that the event $B = \{\text{one of the numbers drawn is } 15\}$ has occurred. You want to calculate the probability of the event A that your ticket is a winner. Both events A and B are expressible in the sample space that consists of the ${}_{30}C_6 = 30!/(6!24!)$ possible combinations of 30 items taken six at a time, namely, the unordered draws of six numbers from $1 - 30$. The event B consists of combinations that include 15. Since there are 29 remaining numbers from which to choose the other five in the winning draw, there are ${}_{29}C_5 = 29!/(5!24!)$ outcomes in B. It follows that

$$\mathbb{P}(B) = \frac{{}_{29}C_5}{{}_{30}C_6} = 0.2.$$

The event A that your ticket is a winner consists of a single outcome that is also in B, so $A \cap B = A$, and

$$\mathbb{P}(A \cap B) = P(A) = \frac{1}{{}_{30}C_6} = 1.68 \times 10^{-6}.$$

It follows that the conditional probability of A given B is

$$\mathbb{P}(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1.68 \times 10^{-6}}{0.2} = 8.4 \times 10^{-6}.$$

This is five times as large as $\mathbb{P}(A)$ before you learned that B had occurred. $\square$

Example 1.3-2 Consider an experiment consists of observing the sum of the dice when two fair dice are thrown; you are then informed that the sum is not greater than 3.

(a) Find the probability of the event that two faces are the same without the given information.

(b) Find the probability of the same event with the information given.

Example 1.3-3 A lot of 100 semiconductor chips contains 20 that are defective. Two chips are selected at random, without replacement, from the lot.

(a) What is the probability that the first one selected is defective?

(b) What is the probability that the second one selected is defective given that the first one was defective?

(c) What is the probability that both are defective?

Example 1.3-4 A number is selected at random from $\{1, 2, 3, ..., 100\}$. Given that the number selected is divisible by 2, find the probability that it is divisible by 3 or 5.

Example 1.3-5 An urn contains four colored balls; two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

Example 1.3-6 Suppose that four balls are selected one at a time, without replacement, from a box containing r red balls and b blue balls ($r \geq 2, b \geq 2$). Determine the probability of obtaining the sequence of outcomes red, blue, red, blue.

Example 1.3-7 For any two events A and B with $\mathbb{P}(B) > 0$, prove that $\mathbb{P}(A'|B) = 1 - \mathbb{P}(A|B)$.

Example 1.3-8 For any three events A, B, and D, such that $\mathbb{P}(D) > 0$, prove that $\mathbb{P}(A \cup B|D) = \mathbb{P}(A|D) + \mathbb{P}(B|D) - \mathbb{P}(A \cap B|D)$.

Exercises from textbook. Section 1.3: 1, 3, 4, 5, 6, 8, 9, 11, 12a, 15, 16.