# Probability and Statistics I

STAT 3600 – Fall 2021

Le Chen
lzc0090@auburn.edu

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Auburn University
Auburn AL

# Chapter 1. Probability

For certain pairs of events, the occurrence of one of them may or may not change the probability of the occurrence of the other. In the latter case, they are said to be independent events. However, before giving the formal definition of independence, let us consider an example.

Example 1.4-1 Flip a fair coin twice and observe the sequence of heads and tails. The sample space is then

$$S = \{HH, HT, TH, TT\}.$$

It is reasonable to assign a probability $1/4$ to each of these four outcomes (equally likely events). Let

$$\begin{aligned} A &= \{\text{heads on the first flip}\} = \{HH, HT\}, \\ B &= \{\text{tails on the second flip}\} = \{HT, TT\}, \\ C &= \{\text{tails on the both flips}\} = \{TT\}. \end{aligned}$$

Then $\mathbb{P}(B) = 2/4 = 1/2$.

Now, on the one hand, if we are given that $C$ has occurred, then $\mathbb{P}(B|C) = 1$, because $C \subset B$. That is, the knowledge of the occurrence of $C$ has changed the probability of $B$.

On the other hand, if we are given that $A$ has occurred, then

$$\mathbb{P}(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{2/4} = \frac{1}{2} = P(B).$$

So the occurrence of $A$ has not changed the probability of $B$. Hence, the probability of $B$ does not depend upon knowledge about event $A$, so we say that $A$ and $B$ are independent events. That is, events $A$ and $B$ are independent if the occurrence of one of them does not affect the probability of the occurrence of the other.

A more mathematical way of saying this is

$$\mathbb{P}(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

provided $\mathbb{P}(A) > 0$ or, in the latte case, $\mathbb{P}(B) > 0$. With the first of these equalities and the multiplication rule, we have

$$\mathbb{P}(A \cap B) = P(A)P(B|A) = P(A)P(B).$$

The second of these equalities, namely, $\mathbb{P}(A|B) = P(A)$, gives us the same result

$$\mathbb{P}(A \cap B) = P(B)P(A|B) = P(B)P(A).$$

Definition 1.4-1 Events $A$ and $B$ are *independent* if and only if $\mathbb{P}(A \cap B) = P(A)P(B)$. Otherwise $A$ and $B$ are called *dependent* events.

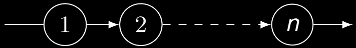
Remark 1.4-1 It follows immediately that if $A$ and $B$ are independent, then

$$\mathbb{P}(A|B) = \mathbb{P}(A) \qquad \text{and} \qquad \mathbb{P}(B|A) = \mathbb{P}(B).$$

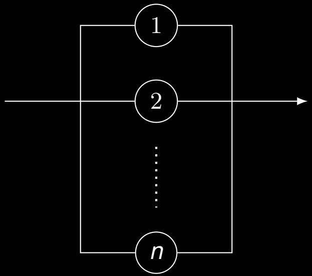
**Theorem 1.4-1** If $A$ and $B$ are independent events, then the following pairs of events are also independent:

(a) $A$ and $B'$. (b) $A'$ and $B$. (c) $A'$ and $B'$.

Example 1.4-2 A system consisting of $n$ separate components is said to be a *series* system if it functions when all $n$ components function. Assume that the components fail independently and that the probability of failure of component $i$ is $p_i$, $i = 1, 2, \cdots, n$. Find the probability that the system functions.



Example 1.4-3 A system consisting of $n$ separate components is said to be a *parallel* system if it functions when at least one of components functions. Assume that the components fail independently and that the probability of failure of component $i$ is $p_i$, $i = 1, 2, \cdots, n$. Find the probability that the system functions.



Example 1.4-4 (Tossing a coin until a head appears) Suppose that a fair coin is tossed until a head appears for the first time, and assume that the outcomes of the tosses are independent. We shall determine the probability $p_n$ that exactly $n$ tosses will be required.

Solution. The desired probability is equal to the probability of obtaining $n-1$ tails in succession and then obtaining a head on the next toss. Since the outcomes of the tosses are independent, the probability of this particular sequence of $n$ outcomes is $p_n = (1/2)^n$. □

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Remark 1.4-2 The probability that a head will be obtained sooner or later (or, equivalently, that tails will not be obtained forever) is

$$\sum_{n=1}^{\infty} p_n = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

Since the sum of the probabilities $p_n$ is 1, it follows that the probability of obtaining an infinite sequence of tails without ever obtaining a head must be 0, which is equivalent to the limit:

$$\lim_{n \to \infty} \left(\frac{1}{2}\right)^n = 0.$$

Definition 1.4-2 (Mutually independence) Events $A$, $B$, and $C$ are *mutually independent* if and only if the following two conditions hold:

(a) $A$, $B$, and $C$ are pairwise independent, that is,

$$\mathbb{P}(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$\mathbb{P}(B \cap C) = P(B)P(C);$$

and (b)

$$\mathbb{P}(A \cap B \cap C) = P(A)P(B)P(C).$$

Example 1.4-5 In the experiment of throwing two fair dice, let $A$ be the event that the first die is odd, $B$ be the event that the second die is odd, and $C$ be the event that the sum is odd. Show that events $A$, $B$, and $C$ are pairwise independent, but $A$, $B$, and $C$ are not independent.

Exercises from textbook: Section 1.4: 1, 2, 3, 5, 7, 8, 9, 11, 12, 13, 16.