# Probability and Statistics I

## STAT 3600 – Fall 2021

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# Chapter 1. Probability

Suppose that we are interested in which of several disjoint events $B_1, \dots, B_k$ will occur and that we will get to observe some other event $A$. If $\mathbb{P}(A|B_i)$ is available for each $i$, then Bayes' theorem is a useful formula for computing the conditional probabilities of the $B_i$ events given $A$, namely, $\mathbb{P}(B_i|A)$.

**Example 1.5-1 (Test for a disease)** Suppose that you are walking down the street and notice that the Department of Public Health is giving a free medical test for a certain disease. The test is 90 percent reliable in the following sense: If a person has the disease, there is a probability of $0.9$ that the test will give a positive response; whereas, if a person does not have the disease, there is a probability of only $0.1$ that the test will give a positive response.

Data indicate that your chances of having the disease are only $1$ in $10,000$. However, since the test costs you nothing, and is fast and harmless, you decide to stop and take the test. A few days later you learn that you had a positive response to the test. Now, what is the probability that you have the disease?

The last question in Example 1.5-1 is a prototype of the question for which Bayes' theorem was designed. We have at least two disjoint events ("you have the disease" and "you do not have the disease") about which we are uncertain, and we learn a piece of information (the result of the test) that tells us something about the uncertain events. Then we need to know how to revise the probabilities of the events in the light of the information we learned.

From multiplication rule $\mathbb{P}(B \cap A) = \mathbb{P}(A|B)P(B)$, we can obtain the following Bayes' rule:

**Theorem 1.5-1 (Bayes' rule)**

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A)} \qquad \text{provided} \qquad \mathbb{P}(A) \neq 0.$$

Let the events $B_1, B_2, ..., B_m$ are mutually exclusive and exhaustive events, that is,

$$S = B_1 \cup B_2 \cup \cdots \cup B_m \qquad \text{and} \qquad B_i \cap B_j = \emptyset,\ i \neq j.$$

Furthermore, suppose the prior probability of the events $B_i$ is positive, that is, $\mathbb{P}(B_i) > 0$, $i = 1, \cdots, m$.

If $A$ is an event, then $A$ is the union of $m$ mutually exclusive events, namely,

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_m \cap A).$$

Thus,

$$\mathbb{P}(A) = \sum_{i=1}^{m} \mathbb{P}(B_i \cap A) = \sum_{i=1}^{m} \mathbb{P}(B_i)\mathbb{P}(A|B_i),$$

which is known as the total probability of event $A$.

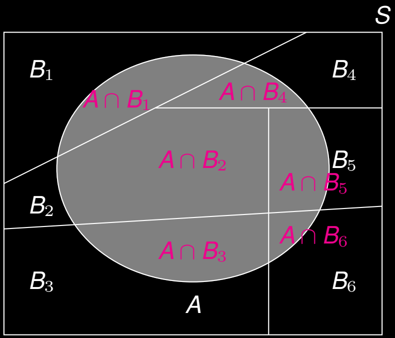
If $\mathbb{P}(A) > 0$, then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(B_k \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\mathbb{P}(A)} \qquad k = 1, 2, \cdots, m.$$

Using total probability equation for $\mathbb{P}(A)$, we obtain

**Theorem 1.5-2 (Bayes' theorem)** Let the events $B_1, B_2, \ldots, B_m$ are mutually exclusive and exhaustive events. If $\mathbb{P}(A) > 0$, then

$$\mathbb{P}(B_k|A) = \frac{\mathbb{P}(B_k)\mathbb{P}(A|B_k)}{\sum_{i=1}^{m} \mathbb{P}(B_i)\mathbb{P}(A|B_i)}, \qquad k = 1, \cdots, m.$$



S
B1
B4
A ∩ B1
A ∩ B4
A ∩ B2
B5
A ∩ B5
B2
A ∩ B6
A ∩ B3
B3
B6
A

Solution(Example 1.5-1). Let us return to the example with which we began this section. We have just received word that we have tested positive for a disease. The test was 90 percent reliable in the sense that we described in Example 1.5-1. We want to know the probability that we have the disease after we learn that the result of the test is positive. Some of you may feel that this probability should be about 0.9. However, this feeling completely ignores the small probability of 0.0001 that you had the disease before taking the test. We shall let $B_1$ denote the event that you have the disease, and let $B_2$ denote the event that you do not have the disease. The events $B_1$ and $B_2$ form a partition. Also, let $A$ denote the event that the response to the test is positive. The event $A$ is information we will learn that tells us something about the partition elements. Then, by Bayes' theorem,

$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \\ &= \frac{(0.9)(0.0001)}{(0.9)(0.0001) + (0.1)(0.9999)} = 0.0009. \end{aligned}$$

$\square$

Remark 1.5-1 Thus, the conditional probability that you have the disease given the test result is approximately only 1 in 1000. Of course, this conditional probability is approxi- mately 9 times as great as the probability was before you were tested, but even the conditional probability is quite small.

Another way to explain this result is as follows: Only one person in every $10,000$ actually has the disease, but the test gives a positive response for approximately one person in every 10. Hence, the number of positive responses is approximately 1000 times the number of persons who actually have the disease. In other words, out of every 1000 persons for whom the test gives a positive response, only one person actually has the disease. This example illustrates not only the use of Bayes' theorem but also the importance of taking into account all of the information available in a problem.

Example 1.5-2 A company producing electric relays has three manufacturing plants producing $50, 30$, and $20$ percent, respectively, of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are $0.02$, $0.05$, and $0.01$, respectively.

(a) If a relay is selected at random from the output of the company, what is the probability that it is defective?

Example 1.5-2 A company producing electric relays has three manufacturing plants producing $50, 30$, and $20$ percent, respectively, of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are $0.02, 0.05$, and $0.01$, respectively.
(b) If a relay selected at random is found to be defective, what is the probability that it was manufactured by plant 2?

Example 1.5-3 (Test for a disease again) There is a new diagnostic test for a disease that occurs in about $0.05\%$ of the population. The test is not perfect, but will detect a person with the disease $99\%$ of the time. It will, however, say that a person without the disease has the disease about $3\%$ of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What are the conditional probabilities that

(a) the person has the disease?

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(b) the person does not have the disease?

Example 1.5-4 In a certain city, 30 percent of the people are Conservatives, 50 percent are Liberals, and 20 percent are Independents. Records show that in a particular election, 65 percent of the Conservatives voted, 82 percent of the Liberals voted, and 50 percent of the Independents voted. If a person in the city is selected at random and it is learned that she did not vote in the last election, what is the probability that she is a Liberal?

Exercises from textbook: Section 1.5: 1, 3, 4, 5, 7, 12.