# Probability and Statistics I

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# Chapter 2. Discrete Distributions

**Definition 2.1-1** Given a random experiment with an outcome space $S$, a function $X$ that assigns one and only one real number $X(s) = x$ to each element $s$ in $S$ is called a *random variable*.

The *space* of $X$ is the set of real numbers $\{x : X(s) = x, s \in S\}$, where $s \in S$ means that the element $s$ belongs to the set $S$.

Example 2.1-1 In the experiment of tossing a coin once, we might define the r.v. X as

$$X(H) = 1 \quad \text{and} \quad X(T) = 0.$$

Note that we could also define another r.v., say $Y$ or $Z$, with

$$Y(H) = 0,\ Y(T) = 1 \qquad \text{or} \qquad Z(H) = 0,\ Z(T) = 0.$$

Example 2.1-2 Consider an experiment in which a person is selected at random from some population and her height in inches is measured. This height is a random variable.

# Events defined by random variables

If $X$ is a r.v. and $x$ is a fixed real number, we can define the event $(X = x)$ as

$$(X = x) = \{s : X(s) = x\} .$$

Similarly, for fixed numbers $x, x_1$,and $x_2$ , we can define the following events:

$$(X \leq x) = \{s : X(s) \leq x\}$$
$$(X > x) = \{s : X(s) > x\}$$
$$(x_1 < X \leq x_2) = \{s : x_1 < X(s) \leq x_2\}$$

These events have probabilities that are denoted by

$$
\begin{aligned}
\mathbb{P}(X = x) &= \mathbb{P}\{s : X(s) = x\} \\
\mathbb{P}(X \leq x) &= \mathbb{P}\{s : X(s) \leq x\} \\
\mathbb{P}(X > x) &= \mathbb{P}\{s : X(s) > x\} \\
\mathbb{P}(x_1 < X \leq x_2) &= \mathbb{P}\{s : x_1 < X(s) \leq x_2\} \\
\mathbb{P}(X \in C) &= \mathbb{P}(s : X(s) \in C).
\end{aligned}
$$

**Definition 2.1-2** Let $X$ be a random variable. The *distribution* of $X$ is the collection of all probabilities of the form $\mathbb{P}(X \in C)$ for all sets $C$ of real numbers such that $\{X \in C\}$ is an event.

Example 2.1-3 In the experiment of tossing a fair coin three times, the sample space $S$, consists of eight equally likely sample points $S = \{HHH, ..., TTT\}$ . If $X$ is the r.v. giving the number of heads obtained, find the distribution of $X$.

Definition 2.1-3 A random variable $X$ is called *discrete* if it takes a finite or countable number (sequence) of values:

$$X \in \{x_1, x_2, x_3, ...\}.$$

It is completely described by telling the probability of each outcome. Distribution defined by:

$$\mathbb{P}(X = x_k) = f(x_k), \quad k = 1, 2, \cdots$$

is called the *probability mass function (p.m.f.)* of the discrete r.v.

Definition 2.1-4 The closure of the set $\{x : f(x) > 0\}$ is called the *support of (the distribution of)* $X$.

Theorem 2.1-1 (Properties of probability mass function) Let $X$ be a discrete r.v. It is p.m.f. $f(x) = \mathbb{P}(X = x)$ satisfies the following properties:

(a) $f(x) \geq 0$, for all $x \in S$;

(b) $\sum_{x \in S} f(x) = 1$;

(c) $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$ where $A \subset S$.

**Definition 2.1-5 (Cumulative distribution function)** We call the function defined by

$$F(x) = \mathbb{P}(X \leq x), \qquad -\infty < x < \infty,$$

the *cumulative distribution function* and abbreviate it as *cdf*.

Definition 2.1-6 When a pmf is constant on the space or support, we say that the distribution is *uniform* over that space.

Example 2.1-4 Let $X$ be a discrete r.v. over a finite number of values $\{1, 2, 3, ..., m\}$ with the pmf:

$$f(x) = \frac{1}{m}, \qquad x = 1, 2, 3, \cdots, m.$$

Then $X$ is uniform over $\{1, 2, 3, ..., m\}$. Its cdf is given by

$$F(x) = \mathbb{P}(X \le x) = \begin{cases} 0, & x < 1, \\ \frac{k}{m}, & k \le x < k+1, \\ 1, & m \le x. \end{cases}$$

This is a (right-continuous) step function. Draw it...

Example 2.1-5 Let

$$f(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that the function $f(x)$ defined by is a pmf of a discrete r.v. $X$.
(b) Find (i) $f(2) = \mathbb{P}(X = 2)$; (ii) $\mathbb{P}(X \leq 2)$; (iii) $\mathbb{P}(X \geq 1)$.

Example 2.1-6 Let $X$ be the number of accidents per week in a factory. Let the pmf of $X$ be

$$f(x) = \frac{1}{(x+1)(x+2)},\ x = 0, 1, 2, \cdots .$$

Find the conditional probability of $X \geq 4$, given that $X \geq 1$.