Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

§ 2.1 Random Variables of the Discrete Type

§ 2.2 Mathematical Expectation

§ 2.3 Special Mathematical Expectation

§ 2.4 The Binomial Distribution

§ 2.5 The Hypergeometric Distribution

§ 2.6 The Negative Binomial Distribution

§ 2.7 The Poisson Distribution

Definition 2.1-1 Given a random experiment with an outcome space S, a function X that assigns one and only one real number $X(s) = x$ to each element s in S is called a *random variable*.
The *space* of X is the set of real numbers $\{x : X(s) = x, s \in S\}$, where $s \in S$ means that the element s belongs to the set S.

Example 2.1-1 In the experiment of tossing a coin once, we might define the r.v. X as

$$X(H) = 1 \quad \text{and} \quad X(T) = 0.$$

Note that we could also define another r.v., say Y or Z, with

$$Y(H) = 0,\ Y(T) = 1 \qquad \text{or} \qquad Z(H) = 0,\ Z(T) = 0.$$

Example 2.1-2 Consider an experiment in which a person is selected at random from some population and her height in inches is measured. This height is a random variable.

Events defined by random variables

If X is a r.v. and x is a fixed real number, we can define the event $(X = x)$ as

$$(X = x) = \{s : X(s) = x\}.$$

Similarly, for fixed numbers x, x_1,and x_2 , we can define the following events:

$$(X \leq x) = \{s : X(s) \leq x\}$$
$$(X > x) = \{s : X(s) > x\}$$
$$(x_1 < X \leq x_2) = \{s : x_1 < X(s) \leq x_2\}$$

These events have probabilities that are denoted by

$$\begin{aligned}
\mathbb{P}(X = x) &= \mathbb{P}\{s : X(s) = x\} \\
\mathbb{P}(X \leq x) &= \mathbb{P}\{s : X(s) \leq x\} \\
\mathbb{P}(X > x) &= \mathbb{P}\{s : X(s) > x\} \\
\mathbb{P}(x_1 < X \leq x_2) &= \mathbb{P}\{s : x_1 < X(s) \leq x_2\} \\
\mathbb{P}(X \in C) &= \mathbb{P}(s : X(s) \in C).
\end{aligned}$$

Definition 2.1-2 Let X be a random variable. The *distribution* of X is the collection of all probabilities of the form $\mathbb{P}(X \in C)$ for all sets C of real numbers such that $\{X \in C\}$ is an event.

Example 2.1-3 In the experiment of tossing a fair coin three times, the sample space S, consists of eight equally likely sample points $S = \{HHH, ..., TTT\}$. If X is the r.v. giving the number of heads obtained, find the distribution of X.

Definition 2.1-3 A random variable X is called *discrete* if it takes a finite or countable number (sequence) of values:

$$X \in \{x_1, x_2, x_3, \ldots\}.$$

It is completely described by telling the probability of each outcome. Distribution defined by:

$$\mathbb{P}(X = x_k) = f(x_k), \quad k = 1, 2, \cdots$$

is called the *probability mass function (p.m.f.)* of the discrete r.v.

Definition 2.1-4 The closure of the set $\{x : f(x) > 0\}$ is called the *support of (the distribution of)* X.

Theorem 2.1-1 (Properties of probability mass function) Let X be a discrete r.v. It is p.m.f. $f(x) = \mathbb{P}(X = x)$ satisfies the following properties:

(a) $f(x) \geq 0$, for all $x \in S$;

(b) $\sum_{x \in S} f(x) = 1$;

(c) $\mathbb{P}(X \in A) = \sum_{x \in A} f(x)$ where $A \subset S$.

Definition 2.1-5 (Cumulative distribution function) We call the function defined by

$$F(x) = \mathbb{P}\left(X \leq x\right), \qquad -\infty < x < \infty,$$

the *cumulative distribution function* and abbreviate it as *cdf*.

Definition 2.1-6 When a pmf is constant on the space or support, we say that the distribution is *uniform* over that space.

Example 2.1-4 Let X be a discrete r.v. over a finite number of values $\{1, 2, 3, ..., m\}$ with the pmf:

$$f(x) = \frac{1}{m}, \qquad x = 1, 2, 3, \cdots, m.$$

Then X is uniform over $\{1, 2, 3, ..., m\}$. Its cdf is given by

$$F(x) = \mathbb{P}(X \leq x) = \begin{cases} 0, & x < 1, \\ \frac{k}{m}, & k \leq x < k+1, \\ 1, & m \leq x. \end{cases}$$

This is a (right-continuous) step function. Draw it...

Example 2.1-5 Let

$$f(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that the function $f(x)$ defined by is a pmf of a discrete r.v. X.

Example 2.1-5 Let

$$f(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find (i) $f(2) = \mathbb{P}(X = 2)$;

Example 2.1-5 Let

$$f(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find (ii) $\mathbb{P}(X \leq 2)$;

Example 2.1-5 Let

$$f(x) = \begin{cases} \frac{3}{4}\left(\frac{1}{4}\right)^x & \text{if } x = 0, 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

(b) Find (iii) $\mathbb{P}(X \geq 1)$.

Example 2.1-6 Let X be the number of accidents per week in a factory. Let the pmf of X be

$$f(x) = \frac{1}{(x+1)(x+2)}, \; x = 0, 1, 2, \cdots .$$

Find the conditional probability of $X \geq 4$, given that $X \geq 1$.