Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

Definition 2.3-1 The *mean (or expected value)* of a discrete r.v. X, denoted by μ or $\mathbb{E}(X)$, is defined by

$$\mu = \mathbb{E}(X) = \sum_{x \in S} x f(x).$$

Example 2.3-1 Say an experiment has probability of success p, where $0 < p < 1$, and probability of failure $q = 1 - p$. This experiment is repeated independently until the first success occurs; say this happen on the X trial. Clearly the space of X is $S_x = \{1, 2, 3, 4, \cdots\}$.

What is $\mathbb{P}(X = x)$, where $x \in S_x$?

We must observe $x - 1$ failures and then a success to have this happen. Thus, due the independence, the probability is

$$f(x) = \mathbb{P}(X = x) = q \cdot q \cdots q \cdot p = q^{x-1}p, \ \ x \in S_X.$$

Since p and q are positive, this is a pmf because

$$\sum_{x \in S_X} q^{x-1}p = \sum_{x=0}^{\infty} q^{x-1}p = \frac{p}{1-q} = \frac{p}{p} = 1.$$

This distribution is called *geometric distribution*.

The mean of this distribution is

$$\mu = \sum_{x=1}^{\infty} xf(x) = (1)p + (2)qp + (3)q^2p + \cdots$$

and

$$q\mu = (q)p + (2)q^2p + (3)q^3p + \cdots .$$

If we subtract these second of these two equations from the first, we have

$$\begin{aligned}(1-q)\mu &= p + pq + pq^2 + pq^3 + \cdots \\ &= \sum_{x=0}^{\infty} pq^x = \frac{p}{1-q} = 1.\end{aligned}$$

That is,

$$\mu = \frac{1}{1-q} = \frac{1}{p}.$$

For illustration, if $p = 1/10$, we would expect $\mu = 10$ trials are needed on average to observe a success. This certainly agrees with out intuition.

Definition 2.3-2 The rth *moment about the origin* of a discrete r.v. X is defined by

$$\mathbb{E}(X^n) = \sum_{x \in S} x^n f(x).$$

Definition 2.3-3 The *variance* of a discrete r.v. X, denoted by σ^2 or $\mathrm{Var}(X)$, is defined by

$$\sigma^2 = \mathrm{Var}(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \sum_{x \in S} (x - \mu)^2 f(x).$$

Definition 2.3-4 The *standard deviation* of a r.v. X, denoted by σ, is the positive square root of $\mathrm{Var}(X)$, i.e.,

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

Example 2.3-2 Consider a discrete r.v. X whose p.m.f. is given by

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } x = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean, variance, and standard deviation of X.

Example 2.3-3 Let a r.v. X denote the outcome of throwing a fair die. Find the mean and variance of X.

Example 2.3-4 Find the mean and variance of a r.v. X, which has a uniform distribution on the first m positive integers.
Hint: Use the formula:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Definition 2.3-5 The *nth moment about the point b* is defined as

$$\mathbb{E}[(X-b)^n] = \sum_{x \in S} (x-b)^n f(x).$$

Remark 2.3-1 $\mathrm{Var}(X)$ is the second moment about the mean μ.

Definition 2.3-6 The *rth factorial moment* is defined as

$$\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)(X-2)\cdots(X-r+1)]$$

Remark 2.3-2

$$\sigma^2 = \mathrm{Var}(X) = \mathbb{E}[X(X-1)] + \mathbb{E}(X) - [\mathbb{E}(X)]^2$$
$$= \mathbb{E}[(X)_2] + \mu - \mu^2.$$

Example 2.3-5 Find the variance of of the hypergeometric distribution considered in Example 2.3-3.

Hint: $\mathbb{E}(X) = n\frac{N_1}{N}$ and $\mathbb{E}[X(X-1)] = \frac{n(n-1)(N_1)(N_1-1)}{N(N-1)}$.

Definition 2.3-7 Let X be a random variable of the discrete type with pmf $f(x)$ and space S. If there is a positive number h such that

$$\mathbb{E}(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < t$, then the function defined by

$$M(t) = \mathbb{E}(e^{tX})$$

is called the *moment-generating function of* X (or of the distribution of X). This function is often abbreviated as mgf.

Properties of moment generating function

1. $M(0) = 1$.

2. If the space of S is $\{b_1, b_2, b_3, \cdots\}$, then the moment generating function is given by the expansion

$$M(t) = e^{tb_1} f(b_1) + e^{tb_2} f(b_2) + e^{tb_3} f(b_3) + \cdots .$$

Thus, the coefficient of e^{tb_i} is the probability

$$f(b_i) = \mathbb{P}(X = b_i).$$

3. If the moment generating function exists, then

$$M'(0) = \mathbb{E}(X) = \mu$$

$$M''(0) = \mathbb{E}(X^2)$$

and, in general,

$$M^{(r)}(0) = \mathbb{E}(X^r)$$

Example 2.3-6 Suppose X has the geometric distribution of Example 2.3-1; that is, the pmf is

$$f(x) = q^{x-1}p, \quad x = 1, 2, 3, \cdots .$$

Find mgf of X.

Example 2.3-7 Define the p.m.f. and give the values of μ, σ^2, and σ when the moment generating function of X is given by (a) $M(t) = 1/3 + (2/3)e^t$; and (b) $M(t) = (0.25 + 0.75e^t)$.

Example 2.3-8 If the moment generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}.$$

Find the pmf, mean, and variance.

Example 2.3-9 Suppose the mgf of X is

$$M(t) = \frac{e^t/2}{1 - e^t/2}, \qquad t < \ln(2).$$

Find the pmf, mean, and variance.

Hint: Use $(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \qquad -1 < z < 1.$

Exercises from textbook: Section 2.3: 1, 2, 3, 4, 5, 8, 9, 11, 13, 19