

Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

§ 2.1 Random Variables of the Discrete Type

§ 2.2 Mathematical Expectation

§ 2.3 Special Mathematical Expectation

§ 2.4 The Binomial Distribution

§ 2.5 The Hypergeometric Distribution

§ 2.6 The Negative Binomial Distribution

§ 2.7 The Poisson Distribution

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§ 2.7 The Poisson Distribution

Definition 2.3-1 The *mean (or expected value)* of a discrete r.v. X , denoted by μ or $\mathbb{E}(X)$, is defined by

$$\mu = \mathbb{E}(X) = \sum_{x \in S} xf(x).$$

Example 2.3-1 Say an experiment has probability of success p , where $0 < p < 1$, and probability of failure $q = 1 - p$. This experiment is repeated independently until the first success occurs; say this happen on the X trial. Clearly the space of X is $S_X = \{1, 2, 3, 4, \dots\}$.

What is $\mathbb{P}(X = x)$, where $x \in S_X$?

We must observe $x - 1$ failures and then a success to have this happen. Thus, due the independence, the probability is

$$f(x) = \mathbb{P}(X = x) = q \cdot q \cdots q \cdot p = q^{x-1} p, \quad x \in S_X.$$

Since p and q are positive, this is a pmf because

$$\sum_{x \in S_X} q^{x-1} p = \sum_{x=0}^{\infty} q^{x-1} p = \frac{p}{1 - q} = \frac{p}{p} = 1.$$

This distribution is called *geometric distribution*.

The mean of this distribution is

$$\mu = \sum_{x=1}^{\infty} xf(x) = (1)p + (2)qp + (3)q^2p + \dots$$

and

$$q\mu = (q)p + (2)q^2p + (3)q^3p + \dots$$

If we subtract these second of these two equations from the first, we have

$$\begin{aligned}(1 - q)\mu &= p + pq + pq^2 + pq^3 + \dots \\ &= \sum_{x=0}^{\infty} pq^x = \frac{p}{1 - q} = 1.\end{aligned}$$

That is,

$$\mu = \frac{1}{1 - q} = \frac{1}{p}.$$

For illustration, if $p = 1/10$, we would expect $\mu = 10$ trials are needed on average to observe a success. This certainly agrees with our intuition.

Definition 2.3-2 The *rth moment about the origin* of a discrete r.v. X is defined by

$$\mathbb{E}(X^n) = \sum_{x \in S} x^n f(x).$$

Definition 2.3-3 The *variance* of a discrete r.v. X , denoted by σ^2 or $\text{Var}(X)$, is defined by

$$\sigma^2 = \text{Var}(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \sum_{x \in S} (x - \mu)^2 f(x).$$

Definition 2.3-4 The *standard deviation* of a r.v. X , denoted by σ , is the positive square root of $\text{Var}(X)$, i.e.,

$$\sigma = \sqrt{\text{Var}(X)}.$$

Example 2.3-2 Consider a discrete r.v. X whose p.m.f. is given by

$$f(x) = \begin{cases} \frac{1}{3} & \text{if } x = -1, 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the mean, variance, and standard deviation of X .

Example 2.3-3 Let a r.v. X denote the outcome of throwing a fair die. Find the mean and variance of X .

Example 2.3-4 Find the mean and variance of a r.v. X , which has a uniform distribution on the first m positive integers.

Hint: Use the formula:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Definition 2.3-5 The n th moment about the point b is defined as

$$\mathbb{E}[(X - b)^n] = \sum_{x \in S} (x - b)^n f(x).$$

Remark 2.3-1 $\text{Var}(X)$ is the second moment about the mean μ .

Definition 2.3-6 The *r*th factorial moment is defined as

$$\mathbb{E}[(X)_r] = \mathbb{E}[X(X-1)(X-2)\cdots(X-r+1)]$$

Remark 2.3-2

$$\begin{aligned}\sigma^2 = \text{Var}(X) &= \mathbb{E}[X(X-1)] + \mathbb{E}(X) - [\mathbb{E}(X)]^2 \\ &= \mathbb{E}[(X)_2] + \mu - \mu^2.\end{aligned}$$

Example 2.3-5 Find the variance of the hypergeometric distribution considered in Example 2.3-3.

Hint: $E(X) = n \frac{N_1}{N}$ and $E[X(X-1)] = \frac{n(n-1)(N_1)(N_1-1)}{N(N-1)}$.

Definition 2.3-7 Let X be a random variable of the discrete type with pmf $f(x)$ and space S . If there is a positive number h such that

$$\mathbb{E}(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function defined by

$$M(t) = \mathbb{E}(e^{tX})$$

is called the *moment-generating function of X* (or of the distribution of X). This function is often abbreviated as mgf.

Properties of moment generating function

1. $M(0) = 1$.

2. If the space of \mathcal{S} is $\{b_1, b_2, b_3, \dots\}$, then the moment generating function is given by the expansion

$$M(t) = e^{tb_1} f(b_1) + e^{tb_2} f(b_2) + e^{tb_3} f(b_3) + \dots$$

Thus, the coefficient of e^{tb_i} is the probability

$$f(b_i) = \mathbb{P}(X = b_i).$$

3. If the moment generating function exists, then

$$M'(0) = \mathbb{E}(X) = \mu$$

$$M''(0) = \mathbb{E}(X^2)$$

and, in general,

$$M^{(r)}(0) = \mathbb{E}(X^r)$$

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Example 2.3-6 Suppose X has the geometric distribution of Example 2.3-1; that is, the pmf is

$$f(x) = q^{x-1}p, \quad x = 1, 2, 3, \dots$$

Find mgf of X .

Example 2.3-7 Define the p.m.f. and give the values of μ , σ^2 , and σ when the moment generating function of X is given by (a) $M(t) = 1/3 + (2/3)e^t$; and (b) $M(t) = (0.25 + 0.75e^t)$.

Example 2.3-8 If the moment generating function of X is

$$M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}.$$

Find the pmf, mean, and variance.

Example 2.3-9 Suppose the mgf of X is

$$M(t) = \frac{e^{t/2}}{1 - e^{t/2}}, \quad t < \ln(2).$$

Find the pmf, mean, and variance.

Hint: Use $(1 - z)^{-1} = 1 + z + z^2 + z^3 + \dots$, $-1 < z < 1$.

Exercises from textbook: Section 2.3: 1, 2, 3, 4, 5, 8, 9, 11, 13, 19