Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

§ 2.1 Random Variables of the Discrete Type

§ 2.2 Mathematical Expectation

§ 2.3 Special Mathematical Expectation

§ 2.4 The Binomial Distribution

§ 2.5 The Hypergeometric Distribution

§ 2.6 The Negative Binomial Distribution

§ 2.7 The Poisson Distribution

Definition 2.4-1 A r.v. X is called a *Bernoulli* r.v. with parameter p if its p.m.f. is given by

$$f(x) = \mathbb{P}(X = x) = p^x (1 - p)^{1-x}, \qquad x = 0 \text{ or } 1,$$

where $0 \leq p \leq 1$.

A Bernoulli r.v. X is associated with some experiment where an outcome can be classified as either a "success" or a "failure," and the probability of a success is p and the probability of a failure is $q = 1 - p$. Such experiments are often called Bernoulli trials.

Theorem 2.4-1 The mean and variance of the Bernoulli r.v. X are

$$\mu = \mathbb{E}(X) = p,$$
$$\sigma^2 = \mathrm{Var}(X) = p(1 - p).$$

Definition 2.4-2 A r.v. X is called a *binomial* r.v. with parameters (n, p) if its pmf is given by

$$f(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, \cdots, n$$

where $0 \leq p \leq 1$ and

$$\binom{n}{x} = \frac{n!}{x!(n-x)!},$$

which is known as the binomial coefficient.

Remark 2.4-1 Recall that if n is positive integer, then

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} b^x a^{n-x}.$$

Thus, if we use binomial expansion, then sum of the binomial probabilities is

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1,$$

a result that had to follow from the fact that $f(x)$ is pmf.

Remark 2.4-2 A binomial r.v. X is associated with some experiments in which n independent Bernoulli trials are performed and X represents the number of successes that occur in the n trials. Note that a Bernoulli r.v. is just a binomial r.v. with parameters $(1, p)$.

We now use the binomial expansion to find the mgf for a binomial random variable and then the mean and variance.

Theorem 2.4-2 Let $X \sim \text{binom}(n, p)$. Then

$$M_X(t) = [(1-p) + pe^t]^n, \quad -\infty < t < \infty.$$

Solution. The mgf is given by

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x} \\ &= [(1-p) + pe^t]^n, \quad -\infty < t < \infty, \end{aligned}$$

from the expansion of $(a+b)^n$ with $a = 1-p$ and $b = pe^t$. □

Theorem 2.4-3 Let $X \sim \text{binom}(n, p)$. Then

$$\mu = np \quad \text{and} \quad \sigma^2 = np(1-p).$$

Solution. By Theorem 2,

$$\mu = \mathbb{E}(X) = M'(0) = n[(1-p) + pe^t]^{n-1}(pe^t)\Big|_{t=0} = np,$$

$$\sigma^2 = \text{Var}(X) = \mathbb{E}[X^2] - (E[X])^2 = M''(0) - [M'(0)]^2 = np(1-p).$$

□

Example 2.4-1 A binary source generates digits 1 and 0 randomly with probabilities 0.6 and 0.4, respectively.
(a) What is the probability that two 1s and three 0s will occur in a five-digit sequence?

Example 2.4-1 A binary source generates digits 1 and 0 randomly with probabilities 0.6 and 0.4, respectively.
(b) What is the probability that at least three 1s will occur in a five-digit sequence?

Example 2.4-2 A fair coin is flipped 10 times. Find the probability of the occurrence of 5 or 6 heads.

Example 2.4-3 For $0 \leq p \leq 1$, and $n = 2, 3, \cdots$, determine the value of

$$\sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}.$$

Exercises from textbook: Section 2.4: 1, 3, 4, 5, 7abc, 10, 17, 20.