Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

Consider a collection of $N = N_1 + N_2$ similar objects, N_1 of them belonging to one of the two dichotomous classes (red chips, say) and N_2 of them belonging to the second class (blue chips, say).

A collection of n objects is selected from these N objects at random and without replacement.

Find the probability that exactly x of these n objects belong to the first class and $n - x$ belong to the second. Clearly, we need

$$0 \leq x \leq N_1 \quad \text{and} \quad 0 \leq n - x \leq N_2, \tag{1}$$

which are equivalent to

$$\max(n - N_2, 0) \leq x \leq \min(n, N_1).$$

We can select x objects from the fist class in any one of $\binom{N_1}{x}$ ways and $n-x$ objects from the second class in any one of $\binom{N_2}{n-x}$ ways.

By multiplication principle, the product $\binom{N_1}{x}\binom{N_2}{n-x}$ equals the number of ways the joint operation can be performed.

If we assume that each of the $\binom{N}{n}$ ways of selecting n objects from $N = N_1 + N_2$ objects has the same probability, it follows that the desired probability is

$$f(x) = \mathbb{P}(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \qquad \max(n - N_2, 0) \leq x \leq \min(n, N_1).$$

Then we say the random variable X has a hypergeometric distribution with parameters N_1, N_2 and n, denoted as $HG(N_1, N_2, n)$.

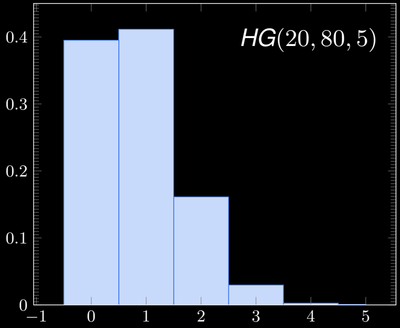
Example 2.5-1 A lot (collection) consisting of 100 fuses is inspected by the following procedure: Five fuses are chased at random and tested; if all five blow at the correct amperage, the lot is accepted. Suppose that the lot contains 20 defective fuses. If X is a random variable equal to the number of defective fuses in the sample of 5, the probability of accepting is

$$\mathbb{P}(X = 0) = \frac{\binom{20}{0}\binom{80}{5}}{\binom{100}{5}} = 0.3193.$$

More generally, the pmf of X is

$$f(x) = \mathbb{P}(X = x) = \frac{\binom{20}{x}\binom{80}{5-x}}{\binom{100}{5}}, \qquad x = 0, 1, 2, 3, 4, 5.$$

x	0	1	2	3	4	5
$f(x)$	$\frac{149380}{378131}$	$\frac{933625}{2268786}$	$\frac{182875}{1134393}$	$\frac{78375}{2646917}$	$\frac{2375}{934206}$	$\frac{38}{467103}$
approx.	0.3951	0.4115	0.1612	0.02961	0.002542	0.00008135



Theorem 2.5-1 Suppose that X follows $HG(N_1, N_2, n)$. Then

$$\mathbb{E}(X) = n\left(\frac{N_1}{N}\right) \qquad \text{and} \qquad \mathrm{Var}(X) = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right).$$

Remark 2.5-1 Check Examples 2.2-3 and 2.3-5.