Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

Let the random variable X denote the number of trials needed to observe the rth success in a sequence of independent Bernoulli trials. That is, X is the trial number on which the rth success is observed.

By the multiplication rule of probabilities, the pmf of X-say, $g(x)$- equals the product of the probability

$$\binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r-1} q^{x-r}$$

of obtaining exactly $r-1$ successes in the first $x-1$ trials and the probability p of success on the rth trial. Thus, the pmf of X is

$$g(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r} q^{x-r}, \quad x = r, r+1, \cdots .$$

We say that X has a negative binomial distribution with parameter (r, p).

Remark 2.6-1 The reason for calling this distribution the negative binomial distribution is as follows:

Consider $h(w) = (1-w)^{-r}$, the binomial $(1-w)$ with the negative exponent $-r$. Using Maclaurin's series expansion, we have

$$(1-w)^{-r} = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} w^k = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} w^k, \quad -1 < w < 1.$$

If we let $x = k + r$ in the summation, then $k = x - r$ and

$$(1-w)^{-r} = \sum_{x=r}^{\infty} \binom{r+x-r-1}{r-1} w^{x-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} w^{x-r},$$

the summand of which is, expect for the factor p^r, the negative binomial probability when $w = q$. In particular, the sum of the probabilities for the negative binomial distribution is 1 because

$$\sum_{x=r}^{\infty} g(x) = \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r q^{x-r} = p^r (1-q)^{-r} = 1.$$

The case $r = 1$

If $r = 1$ in the negative binomial distribution, we note that X has a geometric distribution, since the pmf consists of the term of a geometric series, namely,

$$g(x) = p(1-p)^x, \quad x = 1, 2, 3, \cdots$$

Remark 2.6-2 Recall that for a geometric, the sum is given by

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \text{when } |r| < 1.$$

Thus, for the geometric distribution,

$$\sum_{x=1}^{\infty} g(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1,$$

so that $g(x)$ does satisfy the properties of a pmf.

From the sum of a geometric series, we also note that when k is an integer,

$$\mathbb{P}(X > k) = \sum_{x=k+1}^{\infty} (1-p)^{x-1} p = \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k = q^k.$$

Thus, the value of the cdf at a positive integer k is

$$\mathbb{P}(X \leq k) = \sum_{x=1}^{k} (1-p)^{x-1} p = 1 - P(X > k) = 1 - q^k.$$

General case $r \geq 1$

Theorem 2.6-1 Let X follow a negative binomial distribution with parameters (r, p). Then

$$\mathbb{E}(X) = \frac{r}{p} \quad \text{and} \quad \operatorname{Var}(X) = \frac{rq}{p}, \quad \text{where } q = 1 - p.$$

This theorem is proved by the following example.

Example 2.6-1 Show that the moment generating function of negative binomial random variable X is

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \qquad \text{where } t < -\ln(1-p).$$

Example 2.6-2 Suppose that a sequence of independent tosses are made with a coin for which the probability of obtaining a head on each given toss is $1/30$.
(a) What is the expected number of tosses that will be required in order to obtain five heads?
(b) What is the variance of the number of tosses that will be required in order to obtain five heads?

Remark 2.6-3 Recall that when the moment-generating function exists, derivatives of all orders exist at $t = 0$. Thus, it is possible to represent $M(t)$ as a Maclaurin's series, namely,

$$M(t) = M(0) + M'(0)\left(\frac{t}{1!}\right) + M''(0)\left(\frac{t^2}{2!}\right) + M'''(0)\left(\frac{t^3}{3!}\right) + \cdots$$

Here, $M^{(k)}(0)$ gives the k-th moment.

On the other hand, in many cases, knowing all moments can help us determine the underlying r.v. or distribution.

Example 2.6-3 Let $\mathbb{E}(X^r) = 5^r$, $r = 1, 2, 3, \cdots$. Find the moment-generating function $M(t)$ of X and the pmf of X.

Example 2.6-4 Consider the experiment of throwing a fair dice.
(a) Find the probability that it will take less than six tosses to throw a 6.
(b) Find the probability that it will take more than six tosses to throw a 6.
(c) Find the average number of rolls required in order to obtain a 6.

Exercises form textbook: Section 2.6: 1, 2, 3, 4, 6, 7, 8.