Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 2. Discrete Distributions

Let the random variable X denote the number of trials needed to observe the rth success in a sequence of independent Bernoulli trials. That is, X is the trial number on which the rth success is observed.

By the multiplication rule of probabilities, the pmf of X-say, $g(x)$- equals the product of the probability

$$\binom{x-1}{r-1} p^{r-1}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r-1} q^{x-r}$$

of obtaining exactly $r-1$ successes in the first $x-1$ trials and the probability p of success on the rth trial. Thus, the pmf of X is

$$g(x) = \binom{x-1}{r-1} p^{r}(1-p)^{x-r} = \binom{x-1}{r-1} p^{r} q^{x-r}, \quad x = r, r+1, \cdots.$$

We say that X has a negative binomial distribution with parameter (r, p).

Remark 2.6-1 The reason for calling this distribution the negative binomial distribution is as follows:

Consider $h(w) = (1-w)^{-r}$, the binomial $(1-w)$ with the negative exponent $-r$. Using Maclaurin's series expansion, we have

$$(1-w)^{-r} = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} w^k = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} w^k, \quad -1 < w < 1.$$

If we let $x = k + r$ in the summation, then $k = x - r$ and

$$(1-w)^{-r} = \sum_{x=r}^{\infty} \binom{r+x-r-1}{r-1} w^{x-r} = \sum_{x=r}^{\infty} \binom{x-1}{r-1} w^{x-r},$$

the summand of which is, expect for the factor p^r, the negative binomial probability when $w = q$. In particular, the sum of the probabilities for the negative binomial distribution is 1 because

$$\sum_{x=r}^{\infty} g(x) = \sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r q^{x-r} = p^r (1-q)^{-r} = 1.$$

The case $r = 1$

If $r = 1$ in the negative binomial distribution, we note that X has a geometric distribution, since the pmf consists of the term of a geometric series, namely,

$$g(x) = p(1-p)^x, \quad x = 1, 2, 3, \cdots$$

Remark 2.6-2 Recall that for a geometric, the sum is given by

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \qquad \text{when } |r| < 1.$$

Thus, for the geometric distribution,

$$\sum_{x=1}^{\infty} g(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = \frac{p}{1-(1-p)} = 1,$$

so that $g(x)$ does satisfy the properties of a pmf.

From the sum of a geometric series, we also note that when k is an integer,

$$\mathbb{P}(X > k) = \sum_{x=k+1}^{\infty} (1-p)^{x-1} p = \frac{(1-p)^k p}{1-(1-p)} = (1-p)^k = q^k.$$

Thus, the value of the cdf at a positive integer k is

$$\mathbb{P}(X \leq k) = \sum_{x=1}^{k} (1-p)^{x-1} p = 1 - P(X > k) = 1 - q^k.$$

General case $r \geq 1$

Theorem 2.6-1 Let X follow a negative binomial distribution with parameters (r, p). Then

$$\mathbb{E}(X) = \frac{r}{p} \quad \text{and} \quad \mathrm{Var}(X) = \frac{rq}{p}, \quad \text{where } q = 1 - p.$$

This theorem is proved by the following example.

Example 2.6-1 Show that the moment generating function of negative binomial random variable X is

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \qquad \text{where } t < -\ln(1-p).$$

Example 2.6-2 Suppose that a sequence of independent tosses are made with a coin for which the probability of obtaining a head on each given toss is $1/30$.
(a) What is the expected number of tosses that will be required in order to obtain five heads?

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(b) What is the variance of the number of tosses that will be required in order to obtain five heads?

Remark 2.6-3 Recall that when the moment-generating function exists, derivatives of all orders exist at $t = 0$. Thus, it is possible to represent $M(t)$ as a Maclaurin's series, namely,

$$M(t) = M(0) + M^{'}(0)\left(\frac{t}{1!}\right) + M^{''}(0)\left(\frac{t^2}{2!}\right) + M^{'''}(0)\left(\frac{t^3}{3!}\right) + \cdots$$

Here, $M^{(k)}(0)$ gives the k-th moment.

On the other hand, in many cases, knowing all moments can help us determine the underlying r.v. or distribution.

Example 2.6-3 Let $\mathbb{E}(X^r) = 5^r$, $r = 1, 2, 3, \cdots$. Find the moment-generating function $M(t)$ of X and the pmf of X.

Example 2.6-4 Consider the experiment of throwing a fair dice.
(a) Find the probability that it will take less than six tosses to throw a 6.

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(b) Find the probability that it will take more than six tosses to throw a 6.

Example 2.6-4 Consider the experiment of throwing a fair dice.
(c) Find the average number of rolls required in order to obtain a 6.

Exercises form textbook: Section 2.6: 1, 2, 3, 4, 6, 7, 8.