Probability and Statistics I

STAT 3600 – Fall 2021

Le Chen
lzc0090@auburn.edu

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Auburn University
Auburn AL

Chapter 3. Continuous Distributions

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§ 3.1 Random Variables of the Continuous Type

§ 3.2 The Exponential, Gamma, and Chi-Square Distributions

§ 3.3 The Normal Distributions

§ 3.4 Additional Models

Definition 3.1-1 X is a *continuous* r.v. only if its range contains an interval (either finite or infinite) of real numbers.

Examples of continuous r.v. are the length of time it takes when waiting in line to buy frozen yogurt, the weight of a "1-pound" package of hot dogs, etc.

Definition 3.1-2 The function $f(x)$ is called the *probability density function (pdf)* of the *continuous* r.v. X, with space S if it satisfies the following conditions
(a) $f(x) \geq 0, \forall x \in S$.
(b) $\int_S f(x)dx = 1$.
(c) If $(a, b) \subset S$, then the probability of the event $\{a < X < b\}$ is

$$\mathbb{P}(a < X < b) = \int_a^b f(x)dx.$$

The corresponding distribution of probability is said to be of *continuous type*.

Definition 3.1-3 The *cumulative distribution function (cdf)* of a random variable X of the continuous type, defined in terms of the X, is given by

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)dt, \qquad -\infty < x < \infty.$$

Remark 3.1-1 By the fundamental theorem of calculus, we have, for x values for which the derivative $F'(x)$ exists,

$$F'(x) = f(x).$$

Example 3.1-1 Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} kx & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Determine the value of k and sketch $f(x)$.

Example 3.1-1 Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} kx & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(b) Find and sketch the corresponding cdf $F(x)$.

Example 3.1-1 Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} kx & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.
(c) Find $\mathbb{P}(1/4 < X \leq 2)$.

Remark 3.1-2 Note that if X is continuous r.v.

$$\mathbb{P}(X = b) = 0, \qquad \text{for all real values of } b.$$

As a consequence,

$$\begin{aligned}\mathbb{P}(a \leq X \leq b) &= \mathbb{P}(a < X < b) \\ &= \mathbb{P}(a \leq X < b) \\ &= \mathbb{P}(a < X \leq b) \\ &= F(b) - F(a).\end{aligned}$$

Example 3.1-2 The pdf of a continuous r.v. X is given by

$$f(x) = \begin{cases} 1/3 & \text{if } 0 < x < 1, \\ 2/3 & \text{if } 1 < x < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Find the corresponding cdf $F(x)$ and sketch $f(x)$ and $F(x)$.

Expected value of X, or the mean of X, is

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

The variance of X is

$$\sigma^2 = Var(X) = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx,$$

and the standard deviation of X is

$$\sigma = \sqrt{\mathrm{Var}(X)}.$$

The moment-generating function, if it exists, is

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Example 3.1-3 Find the mean and variance of the r.v. X of Example 3.1-1.

Uniform distribution

Definition 3.1-4 A r.v. X is called a *uniform* r.v. over (a, b) if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding cdf of uniform r.v. X is

$$F(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b. \end{cases}$$

The mean and variance of the uniform r.v. X are

$$\mu = \mathbb{E}(X) = \frac{a+b}{2} \quad \text{and} \quad \sigma^2 = Var(X) = \frac{(b-a)^2}{12}.$$

The moment-generating function is

$$M(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

Example 3.1-4 Let $f(x) = 1/2, -1 \leq x \leq 1$, and 0 otherwise be the pdf of X.
(a) Graph the p.d.f. and distribution function.
(b) Find the mean, variance, and mgf.

Example 3.1-5 If the moment-generating function of X is

$$M(t) = \frac{e^{5t} - e^{4t}}{t}, \quad t \neq 0 \qquad \text{and} \qquad M(0) = 1.$$

Find (a) $\mathbb{E}(X)$; (b) $Var(X)$; and (c) $\mathbb{P}(4.2 < X \leq 4.7)$.

Example 3.1-6 Let X have the pdf

$$f(x) = \begin{cases} xe^{-x} & \text{if } 0 \leq x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Find the mgf, mean, and variance of r.v. X.

Definition 3.1-5 The *(100p)th percentile* is a number π_p such that the area under $f(x)$ to the left of π_p is p, that is,

$$p = \int_{-\infty}^{\pi_p} f(x)dx = F(\pi_p).$$

The 50th percentile is called the *median*. We let $m = \pi_{0.50}$. The 25th and 75th percentiles are called the *first* and and $q_3 = \pi_{0.75}$.

Example 3.1-7

Let X have the pdf

$$f(x) = \begin{cases} e^{-x-1} & -1 < x < \infty, \\ 0 & x \leq -1. \end{cases}$$

(a) Find $\mathbb{P}(X \geq 1)$.

Example 3.1-7

Let X have the pdf

$$f(x) = \begin{cases} e^{-x-1} & -1 < x < \infty, \\ 0 & x \leq -1. \end{cases}$$

(b) Find mgf.

Example 3.1-7

Let X have the pdf

$$f(x) = \begin{cases} e^{-x-1} & -1 < x < \infty, \\ 0 & x \leq -1. \end{cases}$$

(c) Find the mean and variance.

Example 3.1-7

Let X have the pdf

$$f(x) = \begin{cases} e^{-x-1} & -1 < x < \infty, \\ 0 & x \leq -1. \end{cases}$$

(d) Find the first quartile, the second or median, and the third quartile.

Exercises from textbook:section 3.1: 1, 2, 3, 4, 6, 7, 8, 9, 10, 16, 18, 20.