Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 3. Continuous Distributions

Exponential distribution

Definition 3.2-1 A r.v. X is called an *exponential* r.v. with parameter $\lambda(> 0)$ if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & 0 \leq x < \infty, \\ 0 & 0 > x. \end{cases}$$

The corresponding distribution function is

$$F(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ 1 - e^{-x/\theta} & \text{if } 0 \leq x < \infty. \end{cases}$$

and the moment-generating function is

$$M(t) = \frac{1}{1 - \theta t}, \qquad \forall t < \frac{1}{\theta},$$

from which, we see that

$$\mu = M'(0) = \theta \quad \text{and} \quad \sigma^2 = M''(0) - [M'(0)]^2 = \theta^2.$$

Remark 3.2-1 It is useful to note that for an exponential random variable, X, we have

$$\mathbb{P}(X > x) = 1 - F(x) = 1 - (1 - e^{-x/\theta}) = e^{-x/\theta} \quad \text{when} \quad x > 0.$$

Hence, W, the waiting time until the first occurrence in a Poisson process($\lambda > 0$), has an exponential distribution with parameter $\theta = 1/\lambda$.

Example 3.2-1 It is known that the time (in hours) between consecutive traffic accidents can be described by the exponential r.v. X with parameter $\theta = 60$. Find (a) $\mathbb{P}(X \leq 60)$; (b) $\mathbb{P}(X > 120)$; (c) $\mathbb{P}(10 < X < 100)$; (d) the median time.

Example 3.2-2 Suppose that a certain type of electronic component has an exponential distribution with a mean life of 500 hours. If X denotes the life of this component (or the time to failure of this component), then

(a) What is the probability that this component will last at least 300 hours.

Example 3.2-2 Suppose that a certain type of electronic component has an exponential distribution with a mean life of 500 hours. If X denotes the life of this component (or the time to failure of this component), then
(b) Given that it has lasted at least 300 hours, what is the conditional probability that it will last at least another 600 hours.

The most interesting property of the exponential distribution is its "memoryless" property,

$$\mathbb{P}(X > x + t | X > t) = \mathbb{P}(X > x).$$

By this we mean that if the lifetime of an item is exponentially distributed, then an item which has been in use for some hours is as good as a new item with regard to the amount of time remaining until the item fails.

The exponential distribution is the only continuous distribution which possesses this memoryless property.

Gamma distribution

Definition 3.2-2 The *gamma function* is defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \qquad 0 < t.$$

Remark 3.2-2 For $t > 1$, by integration-by-parts, one finds the recursive relation:

$$\Gamma(t) = (t-1)\Gamma(t-1).$$

For example, $\Gamma(5) = 4\Gamma(4)$, $\Gamma(4) = 3\Gamma(3)$, $\Gamma(3) = 2\Gamma(2)$, $\Gamma(2) = 1\Gamma(1)$ and $\Gamma(1) = 1$ Thus, when n is a positive integer, we have

$$\Gamma(n) = (n-1)!$$

Here are some other special values:

$$\begin{aligned}
\Gamma\left(-\frac{3}{2}\right) &= \frac{4\sqrt{\pi}}{3} \\
\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi} \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\
\Gamma(1) &= 0! = 1 \\
\Gamma\left(\frac{3}{2}\right) &= \frac{\sqrt{\pi}}{2} \\
\Gamma(2) &= 1! = 1 \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3\sqrt{\pi}}{4}
\end{aligned}$$

and

$$\frac{1}{\Gamma(-3)} = \frac{1}{\Gamma(-2)} = \frac{1}{\Gamma(-1)} = \frac{1}{\Gamma(0)} = 0$$

Definition 3.2-3 A r.v. X has a *gamma distribution* if its pdf is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty.$$

Remark 3.2-3 W, the waiting time until the αth occurrence in a Poisson process($\lambda > 0$), has a gamma distribution with parameters α and $\theta = 1/\lambda$.

The moment generating function of X is

$$M(t) = \frac{1}{(1-\theta t)^\alpha}, \qquad t < 1/\theta$$

The mean and the variance are

$$\mu = \alpha\theta \quad \text{and} \quad \sigma^2 = \alpha\theta^2.$$

Example 3.2-3 Telephone calls enter a college switchboard at a mean rate of two-thirds of a call per minute according to a Poisson process. Let X denote the waiting time until the tenth call arrives.

(a) What is the pdf of X?

Example 3.2-3 Telephone calls enter a college switchboard at a mean rate of two-thirds of a call per minute according to a Poisson process. Let X denote the waiting time until the tenth call arrives.

(b) What are the moment-generating function, mean, and variance of X.

Example 3.2-4 If X has a gamma distribution with $\theta = 4$ and $\alpha = 2$, find $\mathbb{P}(X < 5)$.

Example 3.2-5 In a medical experiment, a rat has been exposed to some radiation. The experimenters believe that the rat's survival time X (in weeks) has the pdf

$$f(x) = \frac{3x^2}{120^3} e^{-(x/120)^3} \qquad 0 < x < \infty.$$

(a) What is the probability that the rat survives at least 100 weeks? (Ans. $e^{-(125/216)}$)

Example 3.2-5 In a medical experiment, a rat has been exposed to some radiation. The experimenters believe that the rat's survival time X (in weeks) has the pdf

$$f(x) = \frac{3x^2}{120^3} e^{-(x/120)^3} \qquad 0 < x < \infty.$$

(b) Find the expected value of the survival time. (Ans: $120\Gamma(4/3)$)

Chi-square distribution

Definition 3.2-4 A r.v. X has a *chi-square distribution*, denoted as $X \sim \chi^2(r)$, if its pdf is defined by

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-r/2}, \qquad 0 \le x < \infty.$$

Remark 3.2-4 The chi-square distributions is a gamma distribution with $\theta = 2$ and $\alpha = r/2$.

The mean and the variance of this chi-square distribution are, respectively,

$$\mu = \alpha\theta = \left(\frac{r}{2}\right) 2 = r \quad \text{and} \quad \sigma^2 = \alpha\theta^2 = \left(\frac{r}{2}\right) 2^2 = 2r$$

and the moment-generating function is

$$M(t) = (1-2t)^{-r/2} \qquad \forall t < 1/2.$$

Example 3.2-6 If $X \sim \chi^2(23)$, find the following:
(a) $\mathbb{P}(14.85 < X < 32.01)$. (Use Table IV)

Example 3.2-6 If $X \sim \chi^2(23)$, find the following:
(b) Constants a and b such that $\mathbb{P}(a < X < b) = 0.95$ and $\mathbb{P}(X < a) = 0.025$.

Example 3.2-6 If $X \sim \chi^2(23)$, find the following:
(c) The mean and the variance of X.

Example 3.2-6 If $X \sim \chi^2(23)$, find the following:
(d) $\chi^2_{0.05}(23)$ and $\chi^2_{0.95}(23)$.

Exercises from textbook: Section 3.2: 1, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 17, 18, 21, 22