Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 3. Continuous Distributions

§ 3.1 Random Variables of the Continuous Type

§ 3.2 The Exponential, Gamma, and Chi-Square Distributions

§ 3.3 The Normal Distributions

§ 3.4 Additional Models

Definition 3.3-1 A r.v. X follows the *normal distribution* with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, denoted as $X \sim N(\mu, \sigma^2)$, if its pdf is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \qquad -\infty < x < \infty.$$

The moment-generating function of X is

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right).$$

Consequently,

$$\mathbb{E}(X) = M'(0) = \mu,$$

$$\mathrm{Var}(X) = M''(0) - [M'(0)]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

That is, the parameters μ and σ^2 in the pdf of X are the mean and the variance of X.

Standard normal distribution

If $Z \sim N(0,1)$, we say that Z has a standard normal distribution.

The distribution function of Z is

$$\Phi(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2} dw.$$

Values of $\Phi(z)$ for $z \geq 0$ is given in Table Va in the appendix of the textbook.

Because of the symmetry of the standard normal pdf it is true that

$$\Phi(-z) = 1 - \Phi(z), \qquad \forall z \in \mathbb{R}.$$

Again, because of symmetry of the standard normal pdf, when $z > 0$,

$$\Phi(-z) = \mathbb{P}(Z \leq -z) = \mathbb{P}(Z > z)$$

can be read directly from Table Vb.

Example 3.3-1 If $Z \sim N(0, 1)$, find
(a) $\mathbb{P}(0.53 < Z \leq 2.06)$.

Example 3.3-1 If $Z \sim N(0, 1)$, find
(b) $\mathbb{P}(-0.79 \leq Z < 1.52)$.

Example 3.3-1 If $Z \sim N(0, 1)$, find
(c) $\mathbb{P}(Z > -1.77)$.

Example 3.3-1 If $Z \sim N(0, 1)$, find
(d) $\mathbb{P}(Z > 2.89)$.

In statistical applications, we are often interested in finding a number z_α such that

$$\mathbb{P}(Z \geq z_\alpha) = \alpha,$$

where Z is $N(0,1)$ and α is usually less that 0.5. That is, z_α is the $100(1-\alpha)th$ percentile (sometimes called the upper 100α percent point) for standard normal distribution.

Example 3.3-2 Find the values of (a) $z_{0.01}$; (b) $-z_{0.005}$; (c) $z_{0.0475}$.

Theorem 3.3-1 If $X \sim N\left(\mu, \sigma^2\right)$, then $Z = \dfrac{X - \mu}{\sigma} \sim N(0,1)$.

Theorem 3.3-1 can be used to find probabilities relating to $X \sim N\left(\mu, \sigma^2\right)$, as follows:

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

since $(X-\mu)/\sigma$ is $N(0,1)$.

Example 3.3-3 If the moment-generating function of X is $M(t) = \exp(166t + 200t^2)$, find (a) The mean of X; (b) The variance of X; (c) $\mathbb{P}(170 < X < 200)$; (d) $\mathbb{P}(148 < X < 172)$; (e) $\mathbb{P}(|X - 166| < 40)$.

Theorem 3.3-2 If $X \sim N(\mu, \sigma^2)$, then $V = (X - \mu)^2/\sigma^2 = Z^2 \sim \chi^2(1)$.

Example 3.3-4 If Z is $N(0,1)$, find values of c such that (a) $\mathbb{P}(Z \geq c) = 0.025$; (b) $\mathbb{P}(|Z| \leq c) = 0.9$.

Exercises from textbook: Section 3.3: 1, 3, 5cd, 6, 7, 8, 9, 11, 12, 13, 17.