Probability and Statistics I

STAT 3600 – Fall 2021

Le Chen
lzc0090@auburn.edu

Last updated on

July 4, 2021

Auburn University
Auburn AL

Chapter 4. Bivariate Distributions

§ 4.1 Bivariate Distributions of the Discrete Type

§ 4.2 The Correlation Coefficient

§ 4.3 Conditional Distributions

§ 4.4 Bivariate Distributions of the Continuous Type

§ 4.5 The Bivariate Normal Distribution

Definition 4.1-1 Let X and Y be two random variables defined on a discrete probability space. Let S denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type. The probability that $X = x$ and $Y = y$ is denoted by

$$f(x, y) = \mathbb{P}(X = x, Y = y).$$

The function $f(x, y)$ is called the *joint probability mass function (joint pmf)* of X and Y and has the following properties:

(a) $0 \leq f(x, y) \leq 1$.

(b) $\sum \sum_{(x,y)\in S} f(x, y) = 1$.

(c) $\mathbb{P}[(X, Y) \in A] = \sum \sum_{(x,y)\in A} f(x, y)$, where A is a subset of the space S.

Definition 4.1-2 The *joint cdf* of a discrete bivariate r.v. (X, Y) is given by

$$F_{XY}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \sum_{x_i \leq x} \sum_{y_i \leq y} f(x_i, y_i).$$

Example 4.1-1 Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the two tosses and Y is the number of tails that occurs in the two tosses.
(a) What is the range R_X of X?

Example 4.1-1 Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the two tosses and Y is the number of tails that occurs in the two tosses.
(b) What is the range R_Y of Y?

Example 4.1-1 Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the two tosses and Y is the number of tails that occurs in the two tosses.
(c) Find the range R_{XY} of (X, Y).

Example 4.1-1 Consider an experiment of tossing a fair coin twice. Let (X, Y) be a bivariate r.v., where X is the number of heads that occurs in the two tosses and Y is the number of tails that occurs in the two tosses.
(d) Find $\mathbb{P}(X = 2, Y = 0)$, $P(X = 0, Y = 2)$ and $\mathbb{P}(X = 1, Y = 1)$.

Example 4.1-2 Roll a pair of unbiased dice. For each of the 36 sample points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. For example, if the outcome is $(3, 2)$, then the observed values are $X = 2$, $Y = 3$. The event $\{X = 2, Y = 3\}$ could occur in one of two ways - $(3, 2)$ or $(2, 3)$- so its probability is

$$\frac{1}{36} + \frac{1}{36} = \frac{2}{36}.$$

If the outcome is $(2, 2)$, then the observed values are $X = 2$, $Y = 2$. Since the event $\{X = 2, Y = 2\}$ can occur in only one way, $\mathbb{P}(X = 2, Y = 2) = 1/36$. The joint pmf of X and Y is given by the probabilities

$$f(x, y) = \begin{cases} \frac{1}{36}, & \text{if } 1 \leq x = y \leq 6 \\ \frac{2}{36}, & \text{if } 1 \leq x < y \leq 6. \end{cases}$$

when x and y are integers.

Definition 4.1-3 Let X and Y have the joint probability mass function $f(x,y)$ with space S. The probability mass function of X alone, which is called the *marginal probability mass function of X*, is defined by

$$f_1(x) = f_X(x) = \sum_y f(x,y) = \mathbb{P}(X=x), \qquad x \in S_1$$

where the summation is taken over all possible y values for each given x in the x space S_1. That is, the summation is over all (x,y) in S with a given x value.

Similarly, the *marginal probability mass function of Y* is defined by

$$f_2(y) = f_Y(y) = \sum_x f(x,y) = \mathbb{P}(Y=y), \qquad y \in S_2,$$

where the summation is taken over all possible x values for each given y in the y space S_2.

Definition 4.1-4 The random variables X and Y are *independent* if and only if

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y), \qquad \forall x \in S_1, \quad y \in S_2,$$

or equivalently,

$$f(x, y) = f_1(x)f_2(y), \qquad \forall x \in S_1, \quad y \in S_2.$$

Otherwise X and Y are said to be *dependent*.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(a) Find $f_1(x)$, the marginal pmf of X.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(b) Find $f_2(y)$, the marginal pmf of Y.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(c) Find $\mathbb{P}(X > Y)$.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(d) Find $\mathbb{P}(Y = 2X)$.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(e) Find $\mathbb{P}(X + Y = 3)$.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x + y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(f) Find $\mathbb{P}(X \leq 3 - Y)$.

Example 4.1-3 Let the joint pmf of X and Y be defined by

$$f(x, y) = \frac{x+y}{32}, \qquad x = 1, 2, \quad y = 1, 2, 3, 4.$$

(a) Find $f_1(x)$, the marginal pmf of X.
(b) Find $f_2(y)$, the marginal pmf of Y.
(c) Find $\mathbb{P}(X > Y)$.
(d) Find $\mathbb{P}(Y = 2X)$.
(e) Find $\mathbb{P}(X + Y = 3)$.
(f) Find $\mathbb{P}(X \leq 3 - Y)$.
(g) Are X and Y independent or dependent? Why or why not?

Ans.

(a)

(b)

(c)

(d)

(e)

(f)

(g)

Example 4.1-4 The joint pmf of a bivariate r.v. (X, Y) is given by

$$f(x, y) = \begin{cases} k(2x + y), & \text{if } x = 1, 2; y = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.
(a) Find the value of k.

Example 4.1-4 The joint pmf of a bivariate r.v. (X, Y) is given by

$$f(x,y) = \begin{cases} k(2x+y), & \text{if } x = 1,2; y = 1,2 \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.
(b) Find the marginal pmf's of X and Y.

Example 4.1-4 The joint pmf of a bivariate r.v. (X, Y) is given by

$$f(x, y) = \begin{cases} k(2x + y), & \text{if } x = 1, 2; y = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.
(c) Are X and Y independent?

Example 4.1-4 The joint pmf of a bivariate r.v. (X, Y) is given by

$$f(x, y) = \begin{cases} k(2x + y), & \text{if } x = 1, 2; y = 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

where k is a constant.
(a) Find the value of k.
(b) Find the marginal pmf's of X and Y.
(c) Are X and Y independent?
Ans.
(a)
(b)
(c)

Definition 4.1-5 If X_1 and X_2 are random variables of discrete type with the joint pmf $f(x_1, x_2)$ on the space S. If $u(X_1, X_2)$ is a function of these two random variables, then

$$\mathbb{E}[u(X_1, X_2)] = \sum \sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2),$$

if it exists, is called the *mathematical expectation (or expected value)* of $u(X_1, X_2)$.

Example 4.1-5 There are eight chips in a bowl: three marked $(0,0)$, two marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$. A player selects a chip at random and is given the sum of the coordinates in dollars. If X_1 and X_2 represent those two coordinates, respectively, their joint pmf is

$$f(x_1, x_2) = \frac{3 - x_1 - x_2}{8}, \qquad x_1 = 0, 1; x_2 = 0, 1.$$

Find the expected payoff.

Solution. Let $u(X_1, X_2) = \text{payoff} = \$(X_1 + X_2)$. Thus, the expected payoff is given by

$$\begin{aligned} E(u(X_1, X_2)) = \mathbb{E}(X_1 + X_2) &= \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} (x_1 + x_2)\left(\frac{3 - x_1 - x_2}{8}\right) \\ &= (0)\left(\frac{3}{8}\right) + (1)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{1}{8}\right) \\ &= \frac{3}{4}. \end{aligned}$$

That is, the expected payoff is 75 cents. □

Example 4.1-5 There are eight chips in a bowl: three marked $(0,0)$, two marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$. A player selects a chip at random and is given the sum of the coordinates in dollars. If X_1 and X_2 represent those two coordinates, respectively, their joint pmf is

$$f(x_1, x_2) = \frac{3 - x_1 - x_2}{8}, \qquad x_1 = 0, 1; x_2 = 0, 1.$$

Find the expected payoff.

Solution. Let $u(X_1, X_2) = \text{payoff} = \$(X_1 + X_2)$. Thus, the expected payoff is given by

$$\begin{aligned} E(u(X_1, X_2)) = \mathbb{E}(X_1 + X_2) &= \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} (x_1 + x_2)\left(\frac{3 - x_1 - x_2}{8}\right) \\ &= (0)\left(\frac{3}{8}\right) + (1)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{1}{8}\right) \\ &= \frac{3}{4}. \end{aligned}$$

That is, the expected payoff is 75 cents. $\square$

The following mathematical expectations, if they exist, have special names:

(a) If $u_1(X_1, X_2) = X_i$, then

$$\mathbb{E}[u_1(X_1, X_2)] = \mathbb{E}(X_i) = \mu_i$$

is called the mean of $X_i, i = 1, 2$.

(b) If $u_2(X_1, X_2) = (X_i - \mu_i)^2$, then

$$\mathbb{E}[u_2(X_1, X_2)] = \mathbb{E}[(X_i - \mu_i)^2] = \sigma_i^2 = Var(X_i)$$

is called the variance of $X_i, i = 1, 2$.

Remark 4.1-1 The mean μ_i and the variance σ_i^2 can be computed either from the joint pmf $f(x_1, x_2)$ or from the marginal pmf $f_i(x_i), i = 1, 2$.

We now extend the binomial distribution to a trinomial distribution. Here we have three mutually exclusive and exhaustive ways for an experiment to terminate: perfect, "seconds," and defective. We repeat the experiment n independent times, and the probabilities $p_1, p_2, p_3 = 1 - p_1 - p_2$ of perfect, seconds, and defective,respectively, remain the same from trial to trial. In the n trials, let X_1 =number of perfect items, X_2 = number of seconds, and $X_3 = n - X_1 - X_2$ = number of defectives. If x_1 and x_2 are nonnegative integers such that $x_1 + x_2 \leq n$, then the probability of having x_1 perfect, x_2 seconds, and $n - x_1 - x_2$ defectives, in that order, is

$$p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - x_2}.$$

However, if we want $\mathbb{P}(X_1 = x_1, X_2 = x_2)$, then we must recognize that $X_1 = x_1, X_2 = x_2$ can be achieved in

$$\binom{n}{x_1, x_2, n - x_1 - x_2} = \frac{n!}{x_1! x_2! (n - x_1 - x_2)!}$$

different ways. Hence, the **trinomial** pmf is given by

$$\begin{aligned} f(x_1, x_2) &= P(X_1 = x_1, X_2 = x_2) \\ &= \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - x_2}, \end{aligned}$$

where x_1 and x_2 are nonnegative integers such that $x_1 + x_2 \leq n$. Without summing, we know that X_1 is $b(n, p_1)$ and X_2 is $b(n, p_2)$; thus, X_1 and X_2 are dependent, as the product of these marginal probability mass function is not equal to $f(x_1, x_2)$.

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10$, $3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(a) Give the joint pmf of X and Y, $f(x, y)$.

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10, 3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(b) Sketch the set of points for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10, 3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(c) Find $\mathbb{P}(X = 10, Y = 4)$.

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10$, $3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(d) Give the marginal pmf of X.

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10, 3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(e) Find $\mathbb{P}(X \leq 11)$.

Example 4.1-6 A manufactured item is classified as good, a "second," or defective with probability $6/10, 3/10$, and $1/10$, respectively. Fifteen such items are selected at random from the production line. Let X denote the number of good items, Y the number of seconds, and $15 - X - Y$ the number of defective items.
(a) Give the joint pmf of X and Y, $f(x, y)$.
(b) Sketch the set of points for which $f(x, y) > 0$. From the shape of this region, can X and Y be independent? Why or why not?
(c) Find $\mathbb{P}(X = 10, Y = 4)$.
(d) Give the marginal pmf of X.
(e) Find $\mathbb{P}(X \leq 11)$.

Ans.
(a) $f(x, y) = \frac{15!}{x!y!(15-x-y)!}(0.6)^x(0.3)^y(0.1)^{n-x-y}$
(b) no, because the space is not rectangular.
(c) 0.0735.
(d) X is $b(15, 0.6)$.
(e) 0.9095.

Exercises from textbook: 4.1-2, 4.1-3, 4.1-4,4.1-5, 4.1-6, 4.1-8,