Probability and Statistics I

STAT 3600 – Fall 2021

Le Chen
lzc0090@auburn.edu

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Auburn University
Auburn AL

Chapter 4. Bivariate Distributions

Definition 4.2-1 The *covariance* of X and Y, denoted by $\mathrm{Cov}(X, Y)$ or σ_{XY}, is defined by

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

If $\mathrm{Cov}(X, Y) = 0$, then we say that X and Y are *uncorrelated*. X and Y are uncorrelated if and only if

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

Remark 4.2-1 Note that if X and Y are independent, then it can be show that they are uncorrelated, namely,

$$\text{independent} \quad \Rightarrow \quad \text{uncorrelated.}$$

However, the converse is not true in general; that is, the fact that X and Y are uncorrelated does not, in general, imply that they are independent:

$$\text{independent} \quad \not\Leftarrow \quad \text{uncorrelated.}$$

Definition 4.2-2 For two random variables X and Y, the *correlation coefficient*, denoted by ρ_{XY}, is defined by

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Remark 4.2-2 It can be shown that

$$|\rho| \leq 1 \quad \text{or} \quad -1 \leq \rho \leq 1.$$

Remark 4.2-3 Note that the correlation coefficient of X and Y is a measure of linear dependence between X and Y. The *least square regression line* (the line that describes linear relationship between X and Y) is given by

$$y = \mu_Y + \rho_{XY} \frac{\sigma_Y}{\sigma_X} (x - \mu_X).$$

Example 4.2-1 Let X and Y have the joint pmf

$$f(x, y) = \frac{x + y}{32}, \quad x = 1, 2, y = 1, 2, 3, 4.$$

Find the mean μ_X and μ_Y, the variances σ_X^2 and σ_Y^2, the correlation coefficient ρ, and the equation of the least square regression line. Are X and Y independent?

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Ans:
$\mu_X = 25/16$
$\mu_Y = 45/16$
$\sigma_X^2 = 63/256$
$\sigma_Y^2 = 295/256$
$\mathrm{Cov}(X, Y) = -5/256$
$\rho = -0.0367$ dependent.

Example 4.2-2 Let X and Y be random variables of the continuous type having the joint pdf

$$f(x, y) = 2, \quad 0 \leq y \leq x \leq 1.$$

(a) Find the marginal pdf of X and Y.

Example 4.2-2 Let X and Y be random variables of the continuous type having the joint pdf

$$f(x, y) = 2, \quad 0 \le y \le x \le 1.$$

(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.

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$$f(x, y) = 2, \quad 0 \le y \le x \le 1.$$

(c) Determine the equation of the least square regression line.

Example 4.2-2 Let X and Y be random variables of the continuous type having the joint pdf

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(a) Find the marginal pdf of X and Y.
(b) Compute $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \mathrm{Cov}(X, Y)$, and ρ.
(c) Determine the equation of the least square regression line.
Ans:
(a) $f_1(x) = 2x$ for $0 \leq x \leq 1$; $f_2(y) = 2(1 - y)$;
(b) $\mu_X = \mathbb{E}(X) = 2/3$ and $\mu_Y = \mathbb{E}(Y) = 1/3$.

Exercises from textbook: 4.2-1, 4.2-2, 4.2-3, 4.2-7, 4.2-9.