# Probability and Statistics I

## STAT 3600 – Fall 2021

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# Chapter 4. Bivariate Distributions

Definition 4.4-1 *The joint probability density function (joint pdf)* of two continuous-type random variables is an integrable function $f(x, y)$ with the following properties:

(a) $f(x, y) \geq 0$, where $f(x, y) = 0$ only when $(x, y)$ is not in the support (space) $S$ of $X$ and $Y$.

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$.

(c) $\mathbb{P}[(X, Y) \in A] = \iint_A f(x, y)\rangle dxdy$ where $\{(X, Y) \in A\}$ is the event defined in the plane.

Example 4.4-1

The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f(x, y) = \begin{cases} kxy, & \text{if } 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

where $k$ is a constant.

(a) Find the value of $k$.

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(d) Compute $\mu_X; \mu_Y; \sigma_X^2; \sigma_Y^2$.

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$$f(x,y) = \begin{cases} kxy, & \text{if } 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

where $k$ is a constant.
(a) Find the value of $k$.
(b) Are $X$ and $Y$ independent?
(c) Find $\mathbb{P}(X + Y < 1)$
(d) Compute $\mu_X; \mu_Y; \sigma_X^2; \sigma_Y^2$.
Ans.
(a) $k = 4$
(b) they are independent.
(c) $1/6$.

**Definition 4.4-2** If $(X, Y)$ is a continuous bivariate r.v. with joint pdf $f(x, y)$, then the *conditional pdf of $X$*, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x, y)}{f_2(y)}, \quad f_2(y) > 0.$$

Similarly, the *conditional pdf of $Y$*, given that $X = x$, is defined by

$$h(y|x) = \frac{f(x, y)}{f_1(x)}, \quad f_1(x) > 0.$$

Definition 4.4-3 If $(X, Y)$ is a continuous bivariate r.v. with joint pdf $f(x, y)$, the *conditional mean* of $Y$, given that $X = x$ is defined by

$$\mu_{Y|x} = \mathbb{E}(Y|x) = \int_{-\infty}^{\infty} yh(y|x)dy.$$

The *conditional variance* of $Y$, given that $X = x$, is defined by

$$\sigma^2_{Y|x} = Var(Y|x) = \mathbb{E}[(Y - \mu_{Y|x})^2|x] = \int_{-\infty}^{\infty} (y - \mu_{Y|x})^2 h(y|x)dy$$

which can be reduced to

$$\text{Var}(Y|x) = \mathbb{E}(Y^2|x) - (\mu_{Y|x})^2.$$

Example 4.4-2 Let $f(x, y) = 1/40, 0 \leq x \leq 10, 10 - x \leq y \leq 14 - x$, be the joint pdf of $X$ and $Y$.
(a) Find $f_1(x)$, the marginal pdf of $X$.

Example 4.4-2 Let $f(x, y) = 1/40, 0 \le x \le 10, 10 - x \le y \le 14 - x$, be the joint pdf of $X$ and $Y$.
(b) Determine $h(y|x)$, the conditional pdf of $Y$, given that $X = x$.

Example 4.4-2 Let $f(x, y) = 1/40, 0 \leq x \leq 10, 10 - x \leq y \leq 14 - x$, be the joint pdf of $X$ and $Y$.
(c) Calculate $\mathbb{E}(Y|x)$, the conditional mean of $Y$, given that $X = x$.

Example 4.4-2 Let $f(x, y) = 1/40, 0 \leq x \leq 10, 10 - x \leq y \leq 14 - x$, be the joint pdf of $X$ and $Y$.
(d) Find $\mathbb{P}(9 \leq Y \leq 11 | X = 2)$.

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(a) Find $f_1(x)$, the marginal pdf of $X$.

(b) Determine $h(y|x)$, the conditional pdf of $Y$, given that $X = x$.

(c) Calculate $\mathbb{E}(Y|x)$, the conditional mean of $Y$, given that $X = x$.

(d) Find $\mathbb{P}(9 \leq Y \leq 11|X = 2)$.

Ans.

(a) $f_1(x) = 1/10, 0 \leq x \leq 10$;

(b) $h(y|x) = 1/4, 10 - x \leq y \leq 14 - x$ for $0 \leq x \leq 10$;

(c) $\mathbb{E}(Y|x) = 12 - x$.