Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 4. Bivariate Distributions

Definition 4.5-1 A bivariate r.v. (X, Y) is called the *bivariate normal(or gaussian) distribution* if the joint pdf is given by

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left[-\frac{q(x,y)}{2}\right],$$

where

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right].$$

A joint pdf of this form is called a *bivariate normal pdf*.

The marginal pdf of X is

$$f_1(x) = f_X(x) = \frac{1}{\sigma_X\sqrt{2\pi}}\exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right]$$

and so the conditional distribution of Y, given that $X = x$, is a normal distribution with conditional mean

$$\mathbb{E}(Y|x) = \mu_{Y|x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$$

and conditional variance

$$\mathrm{Var}(Y|x) = \sigma_{Y|x}^2 = \sigma_Y^2(1-\rho^2).$$

Example 4.5-1 Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$.
(a) Compute $\mathbb{P}(-5 < X < 5)$.

Example 4.5-1 Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$.
(b) Compute $\mathbb{P}(-5 < X < 5 | Y = 13)$.

Example 4.5-1 Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$.
(c) Compute $\mathbb{P}(7 < Y < 16)$.

Example 4.5-1 Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$.
(d) Compute $\mathbb{P}(7 < Y < 16 | X = 2)$.

Example 4.5-1 Let X and Y have a bivariate normal distribution with parameters $\mu_X = -3, \mu_Y = 10, \sigma_X^2 = 25, \sigma_Y^2 = 9$, and $\rho = 3/5$.
(a) Compute $\mathbb{P}(-5 < X < 5)$.
(b) Compute $\mathbb{P}(-5 < X < 5 | Y = 13)$.
(c) Compute $\mathbb{P}(7 < Y < 16)$.
(d) Compute $\mathbb{P}(7 < Y < 16 | X = 2)$.
Ans.
(a) 0.6006;
(b) 0.7888;
(c) 0.8185;
(d) 0.9371.

Theorem 4.5-1 If X and Y have a bivariate distribution with correlation coefficient ρ, then X and Y are independent if and only if $\rho = 0$.

Exercises from textbook: 4.5-1, 4.5-3, 4.5-6, 4.5-7, 4.5-8.