

# Probability and Statistics I

STAT 3600 – Fall 2021

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# Chapter 5. Distributions of Functions of Random Variables

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§ 5.1 Functions of One Random Variable

§ 5.2 Transformations of Two Random Variables

§ 5.3 Several Random Variables

§ 5.4 The Moment-Generating Function Technique

§ 5.5 Random Functions Associated with Normal Distributions

§ 5.6 The Central Limit Theorem

§ 5.7 Approximations for Discrete Distributions

§ 5.8 Chebyshev Inequality and Convergence in Probability

§ 5.9 Limiting Moment-Generating Functions

Let  $X$  be a random variables of the continuous type. If we consider a function of  $X$ , say  $Y = u(X)$ , then  $Y$  must also be a random variable that has its own distribution. If we can find its distribution function, say,

$$G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(u(X) \leq y),$$

then its pdf is given by  $g(y) = G'(y)$ .

**Example 5.1-1** Let  $X$  have a gamma distribution with pdf

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 < x < \infty,$$

where  $\alpha > 0, \theta > 0$ . Let  $Y = e^X$ , so that the support of  $Y$  is  $1 < y < \infty$ . For each  $y$  in the support, the distribution function of  $Y$  is

$$G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y) = \mathbb{P}(X \leq \ln y).$$

That is,

$$G(y) = \int_0^{\ln y} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx$$

and thus the pdf  $g(y) = G'(y)$  of  $Y$  is

$$g(y) = \frac{1}{\Gamma(\alpha)\theta^\alpha} (\ln y)^{\alpha-1} e^{-(\ln y)/\theta} \left(\frac{1}{y}\right), \quad 1 < y < \infty.$$

Equivalently, we have

$$g(y) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \frac{(\ln y)^{\alpha-1}}{y^{1+1/\theta}} \quad 1 < y < \infty.$$

which is called a *loggamma* pdf.

## Determination of $g(y)$ from $f(x)$

Let  $X$  be a continuous r.v. with pdf  $f(x)$ . If the transformation  $y = u(x)$  is **one-to-one** and has the inverse transformation

$$x = u^{-1}(y) = v(y)$$

then the pdf of  $Y$  is given by

$$g(y) = f(v(y)) |v'(y)|, \quad y \in \mathcal{S}_y,$$

where  $\mathcal{S}_y$  is the support of  $Y$ .

**Example 5.1-2** Let  $Y = 2X + 3$ . Find the pdf of  $Y$  if  $X$  is a uniform r.v. over  $(-1, 2)$ .

$$\text{Ans: } g(y) = \begin{cases} \frac{1}{6}, & \text{if } 1 < y < 7, \\ 0, & \text{otherwise.} \end{cases}$$

**Example 5.1-3** Let  $X$  have the pdf  $f(x) = xe^{-x^2/2}, 0 < x < \infty$ . Find the pdf of  $Y = X^2$ .

Ans:  $g(y) = \dots$



Here are some examples when the transformation  $Y = u(X)$  is not one-to-one.

**Example 5.1-4** Let  $Y = X^2$ . Find the pdf of  $Y$  when the distribution of  $X$  is  $N(0, 1)$ .

Ans.  $g(y) = \frac{1}{\sqrt{2\pi y}} \exp(-y/2)$ ,  $0 < y < \infty$ .

Example 5.1-5 Let  $Y = X^2$ . Find the pdf of  $Y$  when the distribution of  $X$  is

$$f(x) = \frac{x^2}{3}, \quad -1 < x < 2.$$

$$\text{Ans. } g(y) = \begin{cases} \frac{\sqrt{y}}{3}, & \text{if } 0 < y < 1 \\ \frac{\sqrt{y}}{6}, & \text{if } 1 < y < 4. \end{cases}$$

Discuss Theorem 5.1-1 about the simulation of r.v.'s.

Exercises from textbook: Section 5.1: 1, 3, 4ab, 5, 10, 11, 15