Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 5. Distributions of Functions of Random Variables

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Let X be a random variables of the continuous type. If we consider a function of X, say $Y = u(X)$, then Y must also be a random variable that has its own distribution. If we can find its distribution function, say,

$$G(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(u(X) \leq y),$$

then its pdf is given by $g(y) = G'(y)$.

Example 5.1-1 Let X have a gamma distribution with pdf

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 < x < \infty,$$

where $\alpha > 0, \theta > 0$. Let $Y = e^X$, so that the support of Y is $1 < y < \infty$. For each y in the support, the distribution function of Y is

$$G(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \ln y).$$

That is,

$$G(y) = \int_0^{\ln y} \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx$$

and thus the pdf $g(y) = G'(y)$ of Y is

$$g(y) = \frac{1}{\Gamma(\alpha)\theta^\alpha} (\ln y)^{\alpha-1} e^{-(\ln y)/\theta} \left(\frac{1}{y}\right), \qquad 1 < y < \infty.$$

Equivalently, we have

$$g(y) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \frac{(\ln y)^{\alpha-1}}{y^{1+1/\theta}} \qquad 1 < y < \infty.$$

which is called a *loggamma* pdf.

Determination of $g(y)$ from $f(x)$

Let X be a continuous r.v. with pdf $f(x)$. If the transformation $y = u(x)$ is one-to-one and has the inverse transformation

$$x = u^{-1}(y) = v(y)$$

then the pdf of Y is given by

$$g(y) = f\left(v(y)\right)|v'(y)|, \quad y \in S_y,$$

where S_y is the support of Y.

Example 5.1-2 Let $Y = 2X + 3$. Find the pdf of Y if X is a uniform r.v. over $(-1, 2)$.

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Ans: $g(y) = \begin{cases} \frac{1}{6}, & \text{if } 1 < y < 7, \\ 0, & \text{otherwise.} \end{cases}$

Example 5.1-3 Let X have the pdf $f(x) = xe^{-x^2/2}, 0 < x < \infty$. Find the pdf of $Y = X^2$.

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Ans: $g(y) = \cdots$

Here are some examples when the transformation $Y = u(X)$ is not one-to-one.

Example 5.1-4 Let $Y = X^2$. Find the pdf of Y when the distribution of X is $N(0, 1)$.

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Example 5.1-4 Let $Y = X^2$. Find the pdf of Y when the distribution of X is $N(0, 1)$.
Ans. $g(y) = \frac{1}{\sqrt{2\pi y}} \exp(-y/2), \quad 0 < y < \infty.$

Example 5.1-5 Let $Y = X^2$. Find the pdf of Y when the distribution of X is

$$f(x) = \frac{x^2}{3}, \qquad -1 < x < 2.$$

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$$f(x) = \frac{x^2}{3}, \qquad -1 < x < 2.$$

Ans. $g(y) = \begin{cases} \frac{\sqrt{y}}{3}, & \text{if } 0 < y < 1 \\ \frac{\sqrt{y}}{6}, & \text{if } 1 < y < 4.. \end{cases}$

Discuss Theorem 5.1-1 about the simulation of r.v.'s.

Exercises from textbook: Section 5.1: 1, 3, 4ab, 5, 10, 11, 15