Probability and Statistics I

STAT 3600 – Fall 2021

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Chapter 5. Distributions of Functions of Random Variables

Recall that if $X_1, X_2, \cdots, X_n$ are independent, then the joint pdf is the product of the respective pdf's (may not be identically distributed), namely,

$$f(x_1, x_2, \cdots, x_n) = f_1(x_1) f_2(x_2) \cdots f_n(x_n).$$

If they follow the same distribution, then:

Definition 5.3-1 *A random sample of size* n refers a collection of *independent and identically distributed (i.i.d.)* random variables $X1, \cdots, X_n$.

Example 5.3-1 Let X_1 and X_2 be independent Poisson random variables with respective means $\lambda_1 = 2$ and $\lambda_2 = 3$.
(a) Find $\mathbb{P}(X_1 = 3, X_2 = 5)$.

Example 5.3-1 Let X_1 and X_2 be independent Poisson random variables with respective means $\lambda_1 = 2$ and $\lambda_2 = 3$.
(b) Find $\mathbb{P}(X + Y = 1)$.

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(a) Find $\mathbb{P}(X_1 = 3, X_2 = 5)$.
(b) Find $\mathbb{P}(X + Y = 1)$.
Ans. (a) 0.0182 (b) 0.0337

Example 5.3-2 An electronic device runs until one of its three components fails. The lifetime (in weeks), X_1, X_2, X_3, of these components are independent, and each has the Weibull pdf

$$f(x) = \frac{2x}{25} e^{-(x/5)^2}, \qquad 0 < x < \infty.$$

Find the probability that the device stops running in the first three weeks.

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$$f(x) = \frac{2x}{25} e^{-(x/5)^2}, \qquad 0 < x < \infty.$$

Find the probability that the device stops running in the first three weeks.
Ans. 0.660

Theorem 5.3-1 Say $X_1, X_2, \cdots, X_n$ are independent random variables and the random variables $Y = u_1(X_1)u_2(X_2)\cdots u_n(X_n)$. If $\mathbb{E}\left[u_i(X_i)\right], i = 1, 2, \cdots, n$, exists, then

$$\mathbb{E}(Y) = \mathbb{E}\left[u_1(X_1)u_2(X_2)\cdots u_n(X_n)\right] = \mathbb{E}\left[u_1(X_1)\right]\mathbb{E}\left[u_2(X_2)\right]\cdots\mathbb{E}\left[u_n(X_n)\right].$$

Theorem 5.3-2 Let $X_1, X_2, \cdots, X_n$ are independent random variables with respective means $\mu_1, \mu_2 \cdots, \mu_n$ and variances $\sigma_1^2, \sigma_1^2, \cdots, \sigma_n^2$, then the mean and the variance of $Y = \sum_{i=1}^{n} a_i X_i$, where $a_1, a_2, \cdots, a_n$ are real constants, are, respectively,

$$\mu_Y = \sum_{i=1}^{n} a_i \mu_i \quad \text{and} \quad \sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2 .$$

Corollary 5.3-3 Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the distribution with mean μ and variance σ^2. Then the *mean of random sample*

$$\overline{X} := \frac{X_1 + X_2 + \cdots + x_n}{n}$$

has the mean and variance as follows:

$$\mu_{\overline{X}} = \sum_{i=1}^{n} \left(\frac{1}{n}\right) \mu = \mu \quad \text{and} \quad \sigma^2_{\overline{X}} = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2 = \frac{\sigma^2}{n}.$$

Example 5.3-3 Let X_1 and X_2 be a random sample of size $n = 2$ from the exponential distribution with pdf $f(x) = 2e^{-2x}, 0 < x < \infty$.
(a) Find $\mathbb{P}(0.5 < X_1 < 1.0, 0.7 < X_2 < 1.2)$.

Example 5.3-3 Let X_1 and X_2 be a random sample of size $n = 2$ from the exponential distribution with pdf $f(x) = 2e^{-2x}, 0 < x < \infty$.
(b) Find $\mathbb{E}[X_1(X_2 - 0.5)^2]$.

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(a) Find $\mathbb{P}(0.5 < X_1 < 1.0, 0.7 < X_2 < 1.2)$.
(b) Find $\mathbb{E}[X_1(X_2 - 0.5)^2]$.
Ans. (a) ... (b) ...

Example 5.3-4 Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively.
(a) Find $\mathbb{P}(X_1 = 2, X_2 = 2, X_3 = 5)$.

Example 5.3-4 Let X_1, X_2, X_3 be three independent random variables with binomial distributions $b(4, 1/2)$, $b(6, 1/3)$, and $b(12, 1/6)$, respectively.
(b) Find $\mathbb{E}(X_1 X_2 X_3)$.

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(c) Find the mean and the variance of $Y = X_1 + X_2 + X_3$.

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(a) Find $\mathbb{P}(X_1 = 2, X_2 = 2, X_3 = 5)$.
(b) Find $\mathbb{E}(X_1 X_2 X_3)$.
(c) Find the mean and the variance of $Y = X_1 + X_2 + X_3$.
Ans. (a) ... (b) ... (c) ...

Example 5.3-5 Let X_1, X_2, X_3 be independent random variables that represent lifetimes (in hours) of three key components of a device. Say their respective distributions are exponential with means $1000, 1500$, and 2000. Let Y be the minimum of X_1, X_2, X_3 and compute $\mathbb{P}(Y > 1000)$.

Exercises from textbook: Section 5.3: 2, 3, 4, 6, 10, 17, 19.