Probability and Statistics I

STAT 3600 – Fall 2021

Le Chen
lzc0090@auburn.edu

Last updated on

July 4, 2021

Auburn University
Auburn AL

Chapter 5. Distributions of Functions of Random Variables

Theorem 5.5-1 If $X_1, X_2, \cdots, X_n$ are n mutually independent normal variables with means $\mu_1, \mu_2, \cdots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2$, respectively, then the linear function

$$Y = \sum_{i=1}^{n} c_i X_i \sim N\left(\sum_{i=1}^{n} c_i \mu_i, \sum_{i=1}^{n} c_i^2 \sigma_i^2\right).$$

Corollary 5.5-2 If $X_1, X_2, \cdots, X_n$ are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$, then the distribution of the sample mean $\overline{X} = (1/n)\sum_{i=1}^{n} X_i$ follows $N(\mu, \sigma^2/n)$.

Theorem 5.5-3 Let $X_1, X_2, \cdots, X_n$ are observations of a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. Then the *sample mean*

$$\overline{X} := \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the *sample variance*

$$S^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

are independent. Moreover,

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1).$$

Example 5.5-1 Let X equal the widest diameter (in millimeters) of the fetal head measured between the 16th and 25th weeks of pregnancy. Assume that the distribution of X is $N(\mu = 46.58, \sigma^2 = 40.96)$. Let $\overline{X}$ be the sample mean of a random sample of $n = 16$ observations of X.
(a) Give the value of $\mathbb{E}(\overline{X})$ and $\text{Var}(\overline{X})$.
(b) Find $\mathbb{P}(44.42 \leq \overline{X} \leq 48.98)$.
(c) How is $W = \sum_{i=1}^{16} \frac{(X_i - \overline{X})^2}{40.96}$ distributed?
(d) Find $\mathbb{P}[6.262 < W < 30.58]$.
Ans: (a) ... (b) ... (c) ... (d) ...

Theorem 5.5-4 (Student's t distribution) Let

$$T = \frac{Z}{\sqrt{U/r}}$$

where Z is a random variable that is $N(0,1)$, U is a random variable that is $\chi^2(r)$, and Z and U are independent. Then the pdf of T is

$$f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\,\Gamma(r/2)} \frac{1}{(1+t^2/r)^{(r+1)/2}}, \quad -\infty < t < \infty.$$

This distribution is called *Student's t distribution*.

We can use the results of Corollary 5.5-2 and Theorem 5.5-3 and Theorem 5.5-4 to construct an important T random variable. Given a random sample $X_1, X_2, \cdots, X_n$ from a normal distribution, $N(\mu, \sigma^2)$, let

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \quad \text{and} \quad U = \frac{(n-1)S^2}{\sigma^2}.$$

Then the distribution of Z is $N(0, 1)$ by Corollary 5.5-2. Theorem 5.5-3 tells us that the distribution of U is $\chi^2(n-1)$ and that Z and U are independent. Thus,

$$T = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a Student's t distribution with $r = n - 1$ degrees of freedom by Theorem 5.5-4. We use this T to construct confidence intervals for an unknown mean μ of a normal distribution.

Exercises from textbook 5.5-1, 5.5-2, 5.5-3, 5.5-4, 5.5-5, 5.5-6, 5.5-9.