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LINEAR ALGEBRA with Applications

Open Edition

Adapted for

Emory University

Math 221

Linear Algebra

Sections 1 & 2

Lectured and adapted by

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Course page

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ADAPTABLE | ACCESSIBLE | AFFORDABLE

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Supplementary Exercises for Chapter 1

Exercise 1.1 We show in Chapter 4 that the graph of an equation $ax+by+cz=d$ is a plane in space when not all of a, b, and c are zero.

a. By examining the possible positions of planes in space, show that three equations in three variables can have zero, one, or infinitely many solutions.

b. Can two equations in three variables have a unique solution? Give reasons for your answer.

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b. No. If the corresponding planes are parallel and distinct, there is no solution. Otherwise they either coincide or have a whole common line of solutions, that is, at least one parameter.

Exercise 1.2 Find all solutions to the following systems of linear equations.

a.
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + 5x_2 - 2x_3 + x_4 &= 1 \\ -3x_1 - 7x_2 + 7x_3 - 5x_4 &= 7 \\ x_1 + 3x_2 - 4x_3 + 3x_4 &= -5 \end{aligned}$$

b.
$$\begin{aligned} x_1 + 4x_2 - x_3 + x_4 &= 2 \\ 3x_1 + 2x_2 + x_3 + 2x_4 &= 5 \\ x_1 - 6x_2 + 3x_3 &= 1 \\ x_1 + 14x_2 - 5x_3 + 2x_4 &= 3 \end{aligned}$$

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b. $x_1 = \frac{1}{10}(-6s-6t+16)$, $x_2 = \frac{1}{10}(4s-t+1)$, $x_3 = s$, $x_4 = t$

Exercise 1.3 In each case find (if possible) conditions on a, b, and c such that the system has zero, one, or infinitely many solutions.

a)
$$\begin{aligned} x + 2y - 4z &= 4 \\ 3x - y + 13z &= 2 \\ 4x + y + a^2 z &= a+3 \end{aligned}$$

b)
$$\begin{aligned} x + y + 3z &= a \\ ax + y + 5z &= 4 \\ x + ay + 4z &= a \end{aligned}$$

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b. If $a=1$, no solution. If $a=2$, $x=2-2t$, $y=-t$, $z=t$. If $a \neq 1$ and $a \neq 2$, the unique solution is $x = \frac{8-5a}{3(a-1)}$, $y = \frac{-2-a}{3(a-1)}$, $z = \frac{a+2}{3}$

Exercise 1.4 Show that any two rows of a matrix can be interchanged by elementary row transformations of the other two types.

$$\left[\begin{array}{c} R_1 \\ R_2 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1+R_2 \\ R_2 \end{array}\right] \rightarrow \left[\begin{array}{c} R_1+R_2 \\ -R_1 \end{array}\right] \rightarrow \left[\begin{array}{c} R_2 \\ -R_1 \end{array}\right] \rightarrow \left[\begin{array}{c} R_2 \\ R_1 \end{array}\right]$$

Exercise 1.5 If $ad \neq bc$, show that $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ has reduced row-echelon form $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$.

Exercise 1.6 Find a, b, and c so that the system

$$\begin{aligned} x + ay + cz &= 0 \\ bx + cy - 3z &= 1 \\ ax + 2y + bz &= 5 \end{aligned}$$

has the solution $x=3$, $y=-1$, $z=2$.

$a=1$, $b=2$, $c=-1$

Exercise 1.7 Solve the system

$$\begin{aligned} x + 2y + 2z &= -3 \\ 2x + y + z &= -4 \\ x - y + iz &= i \end{aligned}$$

where $i^2=-1$. [See Appendix **??**.]

Exercise 1.8 Show that the *real* system

$$\left\{\begin{aligned} x + y + z &= 5 \\ 2x - y - z &= 1 \\ -3x + 2y + 2z &= 0 \end{aligned}\right.$$

has a *complex* solution: $x=2$, $y=i$, $z=3-i$ where $i^2=-1$. Explain. What happens when such a real system has a unique solution?

The (real) solution is $x = 2$, $y = 3 - t$, $z = t$ where t is a parameter. The given complex solution occurs when $t = 3 - i$ is complex. If the real system has a unique solution, that solution is real because the coefficients and constants are all real.

Exercise 1.9 A man is ordered by his doctor to take 5 units of vitamin A, 13 units of vitamin B, and 23 units of vitamin C each day. Three brands of vitamin pills are available, and the number of units of each vitamin per pill are shown in the accompanying table.

	Vitamin		
Brand	**A**	**B**	**C**
1	1	2	4
2	1	1	3
3	0	1	1

a. Find all combinations of pills that provide exactly the required amount of vitamins (no partial pills allowed).

b. If brands 1, 2, and 3 cost 3¢, 2¢, and 5¢ per pill, respectively, find the least expensive treatment.

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b. 5 of brand 1, 0 of brand 2, 3 of brand 3

Exercise 1.10 A restaurant owner plans to use x tables seating 4, y tables seating 6, and z tables seating 8, for a total of 20 tables. When fully occupied, the tables seat 108 customers. If only half of the x tables, half of the y tables, and one-fourth of the z tables are used, each fully occupied, then 46 customers will be seated. Find x, y, and z.

Exercise 1.11

a. Show that a matrix with two rows and two columns that is in reduced row-echelon form must have one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

[*Hint*: The leading 1 in the first row must be in column 1 or 2 or not exist.]

b. List the seven reduced row-echelon forms for matrices with two rows and three columns.

c. List the four reduced row-echelon forms for matrices with three rows and two columns.

Exercise 1.12 An amusement park charges \$7 for adults, \$2 for youths, and \$0.50 for children. If 150 people enter and pay a total of \$100, find the numbers of adults, youths, and children. [*Hint*: These numbers are nonnegative *integers*.]

Exercise 1.13 Solve the following system of equations for x and y.

$$\begin{aligned} x^2 + \ xy - \ y^2 &= 1 \\ 2x^2 - \ xy + 3y^2 &= 13 \\ x^2 + 3xy + 2y^2 &= 0 \end{aligned}$$

[*Hint*: These equations are linear in the new variables $x_1 = x^2$, $x_2 = xy$, and $x_3 = y^2$.]